TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 270, Number 1, March 1982

ON $K_*(\mathbf{Z}/p^2\mathbf{Z})$ AND RELATED HOMOLOGY GROUPS

BY

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ABSTRACT. It is shown that, for $p \geqslant 5$,

$$R = \mathbf{Z}/p^2\mathbf{Z},\ K_3(R) = \mathbf{Z}/p^2\mathbf{Z} + \mathbf{Z}/(p^2-1)\mathbf{Z}$$

and $K_4(R) = 0$. Similar calculations are made for R the ring of dual numbers over $\mathbf{Z}/p\mathbf{Z}$. The calculation reduces to finding homology groups of $\mathrm{Sl}(R)$. A key tool is the spectral sequence of the group extension of $\mathrm{Sl}(n, p^2)$ over $\mathrm{Sl}(n, p)$. The terms of this spectral sequence depend in turn on the homology of $\mathrm{Gl}(n, p)$ with coefficients various multilinear modules. Calculation of the differentials uses the Charlap-Vasquez description of d^2.

Introduction. In this paper we shall show that, for $p \geqslant 5$,

$$K_3(\mathbf{Z}/p^2\mathbf{Z}) = \mathbf{Z}/p^2\mathbf{Z} \oplus \mathbf{Z}/(p^2-1)\mathbf{Z}$$

and

$$K_4(\mathbf{Z}/p^2\mathbf{Z}) = 0.$$

These groups can be determined from the integral homology groups of $\mathrm{SL}(\mathbf{Z}/p^2\mathbf{Z})$. The prime to p homology has been calculated by Quillen. In this paper, we shall calculate the p-primary component of $H_r(\mathrm{SL}(\mathbf{Z}/p^2\mathbf{Z}), \mathbf{Z})$ for $r = 1, 2, 3, 4$.

Our method is to consider $\mathrm{SL}(n, p^2)$ as an extension of $\mathrm{SL}(n, p)$ and to use the Lyndon-Hochschild-Serre spectral sequence. To determine the E^2 term of the spectral sequence requires rather extensive calculations of homology for $\mathrm{GL}(n, p)$ with coefficients in various modules associated with its multilinear algebra. (For example, we show for $p \geqslant 5$, $n \geqslant 2$, that $H_2(\mathrm{GL}(n, p), V_n \wedge V_n) = \mathbf{Z}/p\mathbf{Z}$ where V_n is the module of $n \times n$ matrices of trace 0.) To calculate the differential d^2 in the spectral sequence, we restrict to the case $n = 2$ and use the theory of Charlap-Vasquez. That theory splits d^2 into two terms, one determined by the group extension and the other by the corresponding split extension. d^2 for this split extension is calculated by cocycle calculations in an explicit double complex.

With minor modifications, our calculations also apply to the split extension $\mathrm{SL}(n, \mathbf{Z}/p\mathbf{Z}[\varepsilon]) \to \mathrm{SL}(n, \mathbf{Z}/p\mathbf{Z})$ where $\mathbf{Z}/p\mathbf{Z}[\varepsilon]$ is the ring of dual numbers. We thus obtain the additional results

$$K_3(\mathbf{Z}/p\mathbf{Z}[\varepsilon]) = \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/(p^2-1)\mathbf{Z},$$
$$K_4(\mathbf{Z}/p\mathbf{Z}[\varepsilon]) = 0 \qquad (p \geqslant 5).$$

Received by the editors February 29, 1980 and, in revised form, July 8, 1980.

1980 *Mathematics Subject Classification*. Primary 18F25; Secondary 20J10, 20J06.

Key words and phrases. K-theory, homology of groups, Lyndon-Hochschild-Serre spectral sequence, Charlap-Vasquez theory, linear groups, dual numbers, $\mathbf{Z}/p^2\mathbf{Z}$.

[1] Partially supported by the National Science Foundation.

We gratefully thank Stewart Priddy and Brian Parshall for many valuable conversations. We also thank Victor Snaith for suggesting in his study of the analogous questions for $p = 2$ that K-groups might be determined by direct computation. Above all, we wish to thank the referee for his careful reading of the paper. In particular, he told us how to remove unnecessary (and annoying) conditions from the crucial Propositions 7.6, 8.1, and 8.4.

Table of notations. p: a prime $\geqslant 5$, unless otherwise noted.

$\mathbf{F} = \mathbf{Z}/p\mathbf{Z}$.

$\mathbf{F}[\varepsilon] = \mathbf{F}[X]/(X^2) =$ the ring of dual numbers over $\mathbf{F}$.

$\mathrm{GL}(n, p) =$ the group of $n \times n$ nonsingular matrices over $\mathbf{Z}/p\mathbf{Z}$. $\mathrm{GL}(n-1, p)$ is imbedded in $\mathrm{GL}(n, p)$ in the upper left-hand square. Similarly, $\mathrm{GL}(n, p^2)$, $\mathrm{SL}(n, p)$, $\mathrm{SL}(n, p^2)$ are defined as usual. $\mathrm{SL}(\mathbf{Z}/p\mathbf{Z})$ and $\mathrm{SL}(\mathbf{Z}/p^2\mathbf{Z})$ are the infinite limits.

$\overline{\mathrm{SL}}$: a related group defined in §2.

M_n: the module of $n \times n$ matrices over $\mathbf{F}$. $\mathrm{GL}(n, p)$ acts by conjugation.

V_n: the submodule of M_n of matrices of trace 0.

$\bigwedge^k A$: the k-fold wedge product over $\mathbf{F}$ of the vector space A.

$\mathfrak{S}^k A$: the k-fold symmetric product over $\mathbf{F}$ of A.

$A \circ A$: the 2-fold symmetric product. $a \circ b$ denotes a typical generator.

$\hat{A} = \mathrm{Hom}_{\mathbf{F}}(A, \mathbf{F})$.

B_n: the subgroup of $\mathrm{GL}(n, p)$ consisting of all upper triangular matrices with nonzero diagonal terms.

U_n: the subgroup of B_n consisting of all upper triangular matrices with 1's on the diagonal.

H_n: the torus subgroup of B_n consisting of all diagonal matrices with nonzero entries.

W_{n-1}: the $n-1$ dimensional vector space $\mathbf{F}^{n-1}$. W_{n-1} is imbedded in B_n as the matrices with 1's on the diagonal and all other entries 0 except those in the last column. Then W_{n-1} is a normal subgroup of U_n and conjugation yields the usual action. Elements u of W_{n-1} are considered column vectors, elements f of $\hat{W}_{n-1}$ are considered row vectors, fu is a scalar, and uf is a matrix.

C_n: the normal subgroup of B_n consisting of all matrices with 1's in the first $n-1$ diagonal positions, a nonzero entry in the last diagonal position, and zeroes elsewhere except in the last column. C_n is the semidirect $\mathbf{F}^* \ltimes W_{n-1}$ where $\mathbf{F}^*$ acts on W_{n-1} by multiplication by the inverse.

e_{ij}: the matrix with 1 in the i, j-position and zeroes elsewhere. The diagonal matrix $(t_1, t_2, \ldots)$ acts on e_{ij} as $t_i t_j^{-1}$.

$h_i = e_{ii} - e_{i+1,i+1}$.

$x_{ij} = I + e_{ij}$.

$[G, A]$: for a G-module A (G a group), the submodule generated by all $g(a) - a$.

$X \otimes_{\mathbf{Z}[V]} A$: X and A are supposed *left* V-modules and $vx \otimes va = x \otimes a$. If V is written additively, $vx \otimes a = x \otimes (-v)a$.

$[v_1, v_2, \ldots, v_r]$: a typical basis element for the standard $Z[V]$-free resolution X of $\mathbf{Z}$. For $r = 0$, $[\cdot]$ denotes the single basis element. $d[u] = u[\cdot] - [\cdot]$ (see §11). On occasion, the image in $X \otimes_{\mathbf{Z}[V]} \mathbf{F}$ is denoted by the same symbol.

$x \cap \zeta$: the Pontryagin product $H_*(V,\mathbf{F}) \otimes H_*(V,\mathbf{F}) \to H_*(V,\mathbf{F})$ induced by the multiplication homomorphism $V \times V \to V$ for V an abelian group. The cup product in $H^*(V,\mathbf{F})$ is denoted simply by multiplication.

δ: $H^t(V,\mathbf{Z}/p\mathbf{Z}) \to H^{t+1}(V,\mathbf{Z}/p\mathbf{Z})$ is the Bockstein morphism, i.e. the connecting homomorphism arising from the exact sequence $0 \to \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p^2\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \to 0$.

$\rho(v)$: the element of $H_2(V,\mathbf{F})$ represented by $\Sigma_{j=0}^{p-1}[v, jv]$. ρ: $V \to H_2(V,\mathbf{F})$ is a monomorphism.

$H_r(\ ,\ ;p)$ the p-primary component of the relevant homology group.

Chapter I. The basic argument

1. Reduction to p-primary homology. From Quillen [**Q**], we know

$$H_r(\mathrm{SL}(\mathbf{Z}/p\mathbf{Z}),\mathbf{Z}) = \begin{cases} 0, & r = 1,2,4, \\ \mathbf{Z}/(p^2-1)\mathbf{Z}, & r = 3. \end{cases}$$

Since for each n, $\mathrm{Ker}\{\mathrm{SL}(n, p^2) \to \mathrm{SL}(n, p)\}$ is a p-group, it follows that

$$H_r(\mathrm{SL}(\mathbf{Z}/p^2\mathbf{Z}),\mathbf{Z}; l) \cong H_r(\mathrm{SL}(\mathbf{Z}/p\mathbf{Z}),\mathbf{Z}; l), \qquad l \neq p.$$

On the other hand, we shall show as a major result of this paper

THEOREM 3.4. *For $p \geqslant 5$,*

$$H_r(\mathrm{SL}(\mathbf{Z}/p^2\mathbf{Z}),\mathbf{Z}; p) = \begin{cases} 0, & r = 1,2,4, \\ \mathbf{Z}/p^2\mathbf{Z}, & r = 3. \end{cases}$$

Hence, we have

COROLLARY 1.1.

$$H_r(\mathrm{SL}(\mathbf{Z}/p^2\mathbf{Z}),\mathbf{Z}) = 0, \qquad r = 1,2,4,$$
$$H_3(\mathrm{SL}(\mathbf{Z}/p^2\mathbf{Z}),\mathbf{Z}) = \mathbf{Z}/p^2\mathbf{Z} \oplus \mathbf{Z}/(p^2-1)\mathbf{Z}.$$

From this we obtain our main result.

THEOREM 1.2. *Let $p \geqslant 5$. Then*

$$K_1(\mathbf{Z}/p^2\mathbf{Z}) = \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/(p-1)\mathbf{Z}, \qquad K_2(\mathbf{Z}/p^2\mathbf{Z}) = 0,$$
$$K_3(\mathbf{Z}/p^2\mathbf{Z}) = \mathbf{Z}/p^2\mathbf{Z} \oplus \mathbf{Z}/(p^2-1)\mathbf{Z}, \qquad K_4(\mathbf{Z}/p^2\mathbf{Z}) = 0.$$

Note. K_2 has been calculated by Milnor [**M**], K_1 by Bass [**B**].

PROOF. We recall that the K-groups of $\mathbf{Z}/p^2$ are defined to be the homotopy groups of $(\mathrm{BGL}\ \mathbf{Z}/p^2)^+$, where $(\ )^+$ is the Quillen plus construction with respect to the (perfect) commutator subgroup of the fundamental group (see [**W**]). By Corollary 1.1, $H_1(\mathrm{BSL}\ \mathbf{Z}/p^2) = 0$ so that $\mathrm{SL}\,\mathbf{Z}/p^2$ is the commutator subgroup of $\mathrm{GL}\,\mathbf{Z}/p^2$. This implies that $(\mathrm{BSL}\,\mathbf{Z}/p^2)^+$ is the universal covering of $(\mathrm{BGL}\,\mathbf{Z}/p^2)^+$ [**W**]. Consequently,

$$K_1(\mathbf{Z}/p^2) = \pi_1((\mathrm{BGL}\,\mathbf{Z}/p^2)^+) = (\mathrm{GL}\,\mathbf{Z}/p^2)/(\mathrm{SL}\,\mathbf{Z}/p^2) \simeq \mathbf{Z}/p-1 \oplus \mathbf{Z}/p,$$
$$K_i(\mathbf{Z}/p^2) \simeq \pi_i((\mathrm{BSL}\,\mathbf{Z}/p^2)^+) \quad \text{for } i > 1.$$

Because $\mathrm{BSL}\,\mathbf{Z}/p^2 \to (\mathrm{BSL}\,\mathbf{Z}/p^2)^+$ is a homology equivalence, Corollary 1.1 and the Hurewicz theorem imply that

$$\pi_2\big((\mathrm{BSL}\,\mathbf{Z}/p^2)^+\big) \simeq H_2(\mathrm{BSL}\,\mathbf{Z}/p^2) = 0,$$
$$\pi_3\big((\mathrm{BSL}\,\mathbf{Z}/p^2)^+\big) \simeq H_3(\mathrm{BSL}\,\mathbf{Z}/p^2) = \mathbf{Z}/p^2 - 1 \oplus \mathbf{Z}/p^2 = \Pi.$$

Moreover, because $(\mathrm{BSL}\,\mathbf{Z}/p^2)^+ \to (\mathrm{BSL}\,\mathbf{Z}/p)^+$ is an l-equivalence of simply connected spaces for primes $l \neq p$, $\pi_4((\mathrm{BSL}\,\mathbf{Z}/p^2)^+\,;\,l) \xrightarrow{\sim} \pi_4((\mathrm{BSL}\,\mathbf{Z}/p)^+\,;\,l)$, and the latter is zero by **[Q]**. Hence, $\pi_4((\mathrm{BSL}\,\mathbf{Z}/p^2)^+) \cong \pi_4((\mathrm{BSL}\,\mathbf{Z}/p^2)^+\,;\,p)$.

Let $(\mathrm{BSL}\,\mathbf{Z}/p^2)^+ \to K(\Pi,3)$ be the natural 3-equivalence obtained by attaching cells to $(\mathrm{BSL}\,\mathbf{Z}/p^2)^+$ to kill the homotopy groups above dimension 3 and let F denote the homotopy fibre of this map. The Hurewicz theorem and the long exact homotopy sequence imply that F is 3-connected and $\pi_4((\mathrm{BSL}\,\mathbf{Z}/p^2)^+) \simeq \pi_4(F) \simeq H_4(F)$. For $p \geqslant 3$, $H_i(K(\Pi,3),\mathbf{Z}/p)$ equals $\mathbf{Z}/p$ for $i = 3,4$ and equals 0 for $i = 5,6$ **[C]**. Consequently, the Hurewicz theorem and the universal coefficient theorem imply that

$$H_3(K(\Pi,3)) \simeq \pi_3(K(\Pi,3)) = \mathbf{Z}/(p^2-1) \oplus \mathbf{Z}/p^2,$$
$$H_i(K(\Pi,3),\mathbf{Z};p) = 0, \qquad 3 < i < 7,\ 3 \leqslant p.$$

Using the Serre spectral sequence

$$E^2_{r,s} = H_r\big(K(\mathbf{Z}/p^2,3),\, H_s(F)\big) \Rightarrow H_{r+s}\big((\mathrm{BSL}\,\mathbf{Z}/p^2)^+\big)$$

we conclude that $H_4(F,\mathbf{Z};\,p) \cong H_4((\mathrm{BSL}\,\mathbf{Z}/p^2)^+,\ \mathbf{Z};\ p) \cong H_4(\mathrm{SL}\,\mathbf{Z}/p^2,\ \mathbf{Z};\ p)$. Consequently, Corollary 1.1 implies that

$$\pi_4\big((\mathrm{BSL}\,\mathbf{Z}/p^2)^+\big) = 0. \quad \square$$

Similar arguments will allow us to conclude for the ring of dual numbers $\mathbf{Z}/p\mathbf{Z}[\varepsilon \mid \varepsilon^2 = 0]$ the following. (Use Theorem 4.1.)

THEOREM 1.3. *Let $p \geqslant 5$. Then*

$$H_r(\mathrm{SL}(\mathbf{F}[\varepsilon]),\mathbf{Z}) = 0, \qquad r = 1,2,4,$$
$$H_3(\mathrm{SL}(\mathbf{F}[\varepsilon]),\mathbf{Z}) = \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/(p^2-1)\mathbf{Z}.$$

Arguing as in the proof of Theorem 1.2, we have

COROLLARY 1.4. *For $p \geqslant 5$*

$$K_1(\mathbf{F}[\varepsilon]) = \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/(p-1)\mathbf{Z}, \qquad K_2(\mathbf{F}[\varepsilon]) = 0,$$
$$K_3(\mathbf{F}[\varepsilon]) = \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/(p^2-1)\mathbf{Z}, \qquad K_4(\mathbf{F}[\varepsilon]) = 0.$$

K_2 was calculated by van der Kallen **[V]**.

2. Reduction to $\overline{\mathrm{SL}}$. Define

$$\overline{\mathrm{SL}}(n,p^2) = \mathrm{Ker}\left\{\mathrm{GL}(n,p^2) \xrightarrow{\det(\ )^{p-1}} U(\mathbf{Z}/p^2\mathbf{Z})\right\}.$$

These groups form a direct limit system, and we define

$$\overline{\mathrm{SL}}(\mathbf{Z}/p^2\mathbf{Z}) = \varinjlim \overline{\mathrm{SL}}(n,p^2).$$

Clearly, $\mathrm{SL}(n, p^2) = \mathrm{Ker}\{\overline{\mathrm{SL}}(n, p^2) \xrightarrow{\det(\,)} U(\mathbf{Z}/p^2\mathbf{Z}) \to U(\mathbf{Z}/p\mathbf{Z})\}$. Moreover, for $(n, p-1) = 1$,

$$\overline{\mathrm{SL}}(n, p^2) = \mathrm{SL}(n, p^2) \times \mathbf{Z}/(p-1)\mathbf{Z},$$

so $H_r(\mathrm{SL}(n, p^2), \mathbf{Z}; p) \xrightarrow{\cong} H_r(\overline{\mathrm{SL}}(n, p^2), \mathbf{Z}; p)$, and by direct limit

$$H_r(\mathrm{SL}(\mathbf{Z}/p^2\mathbf{Z}), \mathbf{Z}; p) \cong H_r(\overline{\mathrm{SL}}(\mathbf{Z}/p^2\mathbf{Z}), \mathbf{Z}; p).$$

In what follows, we calculate the right-hand groups for $r = 1, 2, 3, 4$.

3. Calculation of $H_r(\overline{\mathrm{SL}}(n, p^2), \mathbf{Z}; p)$, $r = 1, 2, 3, 4$, $p \geqslant 5$; the basic argument. The homomorphism π: $\mathbf{Z}/p^2\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z}$ induces an epimorphism π: $\overline{\mathrm{SL}}(n, p^2) \to \mathrm{GL}(n, p)$. Its kernel consists of all $n \times n$ matrices of the form $I + pA$ where $A \in M_n(\mathbf{Z}/p^2\mathbf{Z})$ and $\det(I + pA)^{p-1} = 1$. However, $\det(I + pA)^{p-1} = 1 + (p-1)\mathrm{Tr}(pA) = 1$ if and only if $\mathrm{Tr}(pA) = 0$. Since $I + pA_1 = I + pA_2$ if and only if $A_1 \equiv A_2 \bmod p$, $I + pA \mapsto \pi(A)$ identifies the kernel with $V_n = \mathrm{sl}(n, p) =$ the additive group of $n \times n$ matrices over $\mathbf{Z}/p\mathbf{Z}$ with trace 0. $\mathrm{GL}(n, p)$ acts on the kernel by conjugation x: $T \mapsto xTx^{-1}$.

To calculate the desired homology, we use the spectral sequence of the group extension described above, or rather its p-primary component

$$H_s(\mathrm{GL}(n, p), H_t(V_n, \mathbf{Z}); p) \Rightarrow H_{s+t}(\overline{\mathrm{SL}}(n, p^2), \mathbf{Z}; p) \qquad (s + t > 0).$$

Note that since all the relevant groups are finite, the spectral sequence does decompose into independent primary components. Also, for $t > 0$, we may omit the restriction to p-primary component since $H_t(V_n, \mathbf{Z})$ is a p-group.

We list here the relevant homology groups of V_n. The decompositions are as $\mathrm{GL}(n, p)$-modules and hold for $p > 2$ (but not $p = 2$). Also, they are natural with respect to n. (These facts are easily established using the known structure of $H_*(V_n, \mathbf{Z}/p\mathbf{Z})$ for V_n an elementary abelian p-group.)

$$H_0(V_n, \mathbf{Z}) = \mathbf{Z}, \quad H_1(V_n, \mathbf{Z}) = V_n, \quad H_2(V_n, \mathbf{Z}) = \wedge^2 V_n.$$

(H_2 is spanned by all Pontryagin products of elements of $H_1(V_n, \mathbf{Z})$.)

$$H_3(V_n, \mathbf{Z}) = \wedge^3 V_n \oplus \mathbb{S}^2 V_n,$$
$$H_4(V_n, \mathbf{Z}) = \wedge^4 V_n \oplus \mathrm{Ker}\{V_n \otimes \wedge^2 V_n \to \wedge^3 V_n\}.$$

PROPOSITION 3.0. *We have the following basic table of results for $n \geqslant 2$, $p \geqslant 5$.*

(a) $H_s(\mathrm{GL}(n, p), \mathbf{Z}; p) = H_s(\mathrm{GL}(n, p), \mathbf{Z}/p\mathbf{Z}) = 0$, $1 \leqslant s \leqslant 2p - 4$.

(b)

$$H_s(\mathrm{GL}(n, p), V_n) = \begin{cases} 0, & s = 0, 1, 3, \\ \mathbf{Z}/p\mathbf{Z}, & s = 2 \ (\textit{stably, i.e. } H(\mathrm{GL}(2, p), V_2) \\ & \quad \to H_2(\mathrm{GL}(n, p), V_n) \textit{ is an isomorphism}). \end{cases}$$

(c)

$$H_s(\mathrm{GL}(n, p), \wedge^2 V_n) = \begin{cases} 0, & s = 0, 1, \\ \mathbf{Z}/p\mathbf{Z}, & s = 2. \end{cases}$$

(d)

$$H_s(\mathrm{GL}(n,p), S^2V_n) = \begin{cases} \mathbf{Z}/p\mathbf{Z}, & s = 0\ (\textit{stably}), \\ 0, & s = 1. \end{cases}$$

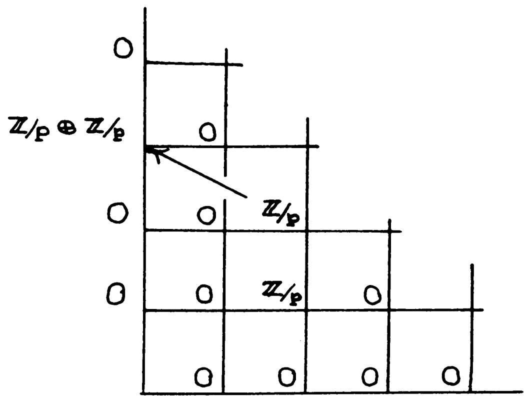
(e)

$$H_s(\mathrm{GL}(n,p), \wedge^3 V_n) = \begin{cases} \mathbf{Z}/p\mathbf{Z}, & s = 0\ (\textit{stably}), \\ 0, & s = 1. \end{cases}$$

(f) $H_0(\mathrm{GL}(n,p), V_n \otimes \wedge^2 V_n) = \mathbf{Z}/p\mathbf{Z}$ (*stably*).
(g) $H_0(\mathrm{GL}(n,p), \wedge^4 V_n) = 0$.

The *proof* of this proposition takes up the major part of this paper. See Chapter II, Propositions 5.1, 5.2, 7.1, 7.3, 7.5, 7.6, and 8.1–8.4.

Using this basic table, we may conclude that the E^2 term of the spectral sequence looks like:



In more detail, we have

PROPOSITION 3.1. ($p \geqslant 5, n \geqslant 2$.)
(i) $H_s(\mathrm{GL}(n,p), H_0(V_n, \mathbf{Z}); p) = 0$, $s = 1, 2, 3, 4$.
(ii)

$$H_s(\mathrm{GL}(n,p), H_1(V_n, \mathbf{Z})) = \begin{cases} 0, & s = 0, 1, 3, \\ \mathbf{Z}/p\mathbf{Z}, & s = 2\ (\textit{stably}). \end{cases}$$

(iii)

$$H_s(\mathrm{GL}(n,p), H_2(V_n, \mathbf{Z})) = \begin{cases} 0, & s = 0, 1, \\ \mathbf{Z}/p\mathbf{Z}, & s = 2. \end{cases}$$

(iv)

$$H_s(\mathrm{GL}(n,p), H_3(V_n, \mathbf{Z})) = \begin{cases} \mathbf{Z}/p \oplus \mathbf{Z}/p, & s = 0\ (\textit{stably}) \\ 0, & s = 1. \end{cases}$$

(v) $H_0(\mathrm{GL}(n,p), H_4(V_n, \mathbf{Z})) = 0$.

PROOF. (i) is a restatement of (a) in the *basic table*. (ii) is (b) and (iii) is (c). (iv) follows from (d) and (e). To prove (v), consider the exact sequence $0 \to \mathrm{Ker} \to V_n \otimes \wedge^2 V_n \to \wedge^3 V_n \to 0$. Since $H_1(\mathrm{GL}(n, p), \wedge^3 V_n) = 0$ by (e), we obtain $H_0(\mathrm{GL}(n, p), \mathrm{Ker}) = 0$ from:

$$\begin{array}{ccccccc} 0 \to H_0(\mathrm{GL}, \mathrm{Ker}) & \to & H_0(\mathrm{GL}, V_n \otimes \wedge^2 V_n) & \to & H_0(\mathrm{GL}, \wedge^3 V_n) & \to 0 \\ & & \| & & \| & \\ & & \mathbf{Z}/p\mathbf{Z} & & \mathbf{Z}/p\mathbf{Z} & \end{array}$$

Since $H_0(\mathrm{GL}, \wedge^4 V) = 0$ by (g), the desired result follows. □

To complete the calculation, we must investigate the differential:

$$\begin{array}{cccc} d^2_{2,2}\colon & E^2_{2,2}(\mathbf{Z}) & \to & E^2_{0,3}(\mathbf{Z}) \\ & \| & & \| \\ & \mathbf{Z}/p\mathbf{Z} & & \mathbf{Z}/p \oplus \mathbf{Z}/p \end{array}$$

We first consider the case $n = 2$. From the basic table (c), for $n = 2$, $E^2_{2,2}(\mathbf{Z}) = \mathbf{Z}/p$. Moreover, we have

PROPOSITION 12.4(b). *For $n = 2$, $p > 3$, and coefficient module $\mathbf{Z}$, $d^2_{2,2}$ is a monomorphism.*

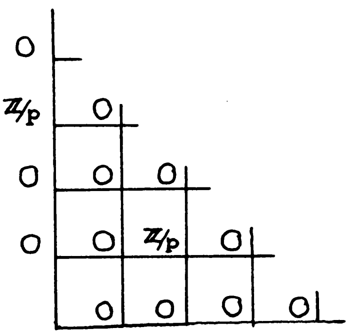
The proof of this fact is based on the Charlap-Vasquez theory and is discussed at length in Chapter III.

To deal with $n \geqslant 2$, consider the commutative diagram:

$$\begin{array}{ccc} H_0(\mathrm{GL}(n, p), H_3(V_n, \mathbf{Z})) & \overset{d^2_{2,2}}{\leftarrow} & H_2(\mathrm{GL}(n, p), H_2(V_n, \mathbf{Z})) = \mathbf{Z}/p\mathbf{Z} \\ \uparrow\cong & & \uparrow \\ H_0(\mathrm{GL}(2, p), H_3(V_2, \mathbf{Z})) & \underset{\text{mono}}{\overset{d^2_{2,2}}{\leftarrow}} & H_2(\mathrm{GL}(n, 2), H_2(V_2, \mathbf{Z})) = \mathbf{Z}/p\mathbf{Z} \end{array}$$

We conclude that $d^2_{2,2}$ is a monomorphism for $n \geqslant 2$.

To continue with the basic argument, we can now conclude that the E^3 term of the spectral sequence looks like:



It follows that for total homology $H_1 = 0$, $H_2 = 0$, $H_3 = \mathbf{Z}/p^2$ or $\mathbf{Z}/p \oplus \mathbf{Z}/p$, $H_4 = 0$.

To determine precisely what happens for H_3, we must repeat the above argument for coefficient group $\mathbf{Z}/p\mathbf{Z}$. We have

$$H_s(\mathrm{GL}(n,\,p),\,H_t(V_n,\mathbf{Z}/p\mathbf{Z})) \Rightarrow H_{s+t}(\overline{\mathrm{SL}}(n,\,p^2),\mathbf{Z}/p\mathbf{Z}).$$

Also, for $p > 2$, we have decompositions of GL-modules

$$\begin{aligned} H_0(V_n,\mathbf{Z}/p\mathbf{Z}) &= \mathbf{Z}/p\mathbf{Z},\\ H_1(V_n,\mathbf{Z}/p\mathbf{Z}) &= V_n,\\ H_2(V_n,\mathbf{Z}/p\mathbf{Z}) &= V_n \oplus \wedge^2 V_n,\\ H_3(V_n,\mathbf{Z}/p\mathbf{Z}) &= \wedge^3 V_n \oplus V_n \otimes V_n = \wedge^3 V_n \oplus \wedge^2 V_n \oplus \mathbb{S}^2 V_n. \end{aligned}$$

V_n is imbedded in $H_2(V_n,\mathbf{Z}/p\mathbf{Z})$ as all $\rho(v)$, $v \in V_n$, where $\rho(v)$ is represented by the chain $\sum_{i=0}^{p-1}[v,\,iv]$.

$\wedge^2 V_n$ is imbedded in $H_2(V_n,\mathbf{Z}/p\mathbf{Z})$ as Pontryagin products $u \cap v$, $u, v \in V_n$, and similarly for $\wedge^3 V_n$ in $H_3(V_n,\mathbf{Z}/p\mathbf{Z})$.

$V_n \otimes V_n$ is imbedded in H_3 as all $u \cap \rho(v)$.

Hence, from the *basic table*, we have

PROPOSITION 3.2. *For $p \geq 5$, $n \geq 2$,*

$$H_s(\mathrm{GL}(n,\,p),\,H_0(V_n,\mathbf{Z}/p\mathbf{Z})) = 0, \qquad s = 1,2,3,4;$$

$$H_s(\mathrm{GL}(n,\,p),\,H_1(V_n,\mathbf{Z}/p\mathbf{Z})) = \begin{cases} 0, & s = 0,1,3,\\ \mathbf{Z}/p\mathbf{Z}, & s = 2; \end{cases}$$

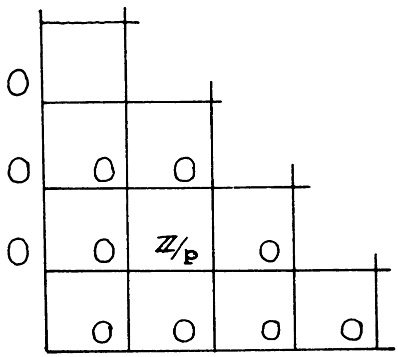
$$H_s(\mathrm{GL}(n,\,p),\,H_2(V_n,\mathbf{Z}/p)) = \begin{cases} 0, & s = 0,1,\\ \mathbf{Z}/p \oplus \mathbf{Z}/p, & s = 2; \end{cases}$$

$$H_0(\mathrm{GL}(n,\,p),\,H_3(V_n,\mathbf{Z}/p)) = \mathbf{Z}/p \oplus \mathbf{Z}/p \quad (\textit{stably}).$$

Again, we will show in Chapter III

PROPOSITION 12.3(a). $d^2_{2,2}\colon E^2_{2,2}(\mathbf{Z}/p\mathbf{Z}) \to E^2_{0,3}(\mathbf{Z}/p\mathbf{Z})$ *is an isomorphism for $n = 2$.*

It follows as before that $d^2_{2,2}$ is an isomorphism for $\mathbf{Z}/p\mathbf{Z}$ for $n \geq 2$. Hence, $E^3(\mathbf{Z}/p\mathbf{Z})$ looks like:



It follows that $H_3(\overline{\mathrm{SL}}(n,\,p^2),\mathbf{Z}/p\mathbf{Z}) = \mathbf{Z}/p\mathbf{Z}$. Hence, by the universal coefficient theorem we may conclude $H_3(\overline{\mathrm{SL}}(n,\,p^2),\mathbf{Z};\,p) = \mathbf{Z}/p^2$. $\square$

To summarize, we may now state one of our basic results.

THEOREM 3.3. *For $p \geqslant 5$, $n \geqslant 2$, we have stably*

$$H_1(\overline{\mathrm{SL}}(n, p^2), \mathbf{Z}; p) = 0,$$
$$H_2(\overline{\mathrm{SL}}(n, p^2), \mathbf{Z}; p) = 0,$$
$$H_3(\overline{\mathrm{SL}}(n, p^2), \mathbf{Z}; p) = \mathbf{Z}/p^2\mathbf{Z},$$
$$H_4(\overline{\mathrm{SL}}(n, p^2), \mathbf{Z}; p) = 0.$$

PROOF. The only thing to check is stability for degree 3. However, we know comparison with $n = 2$ yields isomorphisms for $E^2_{s,t}(\mathbf{Z})$ in all relevant positions. Thus, we conclude that comparison with $n = 2$ yields an isomorphism also for the relevant total homology. □

We have now proved

THEOREM 3.4. *For $p \geqslant 5$,*

$$H_r(\mathrm{SL}(\mathbf{Z}/p^2), \mathbf{Z}; p) = \begin{cases} 0, & r = 1, 2, 4, \\ \mathbf{Z}/p^2\mathbf{Z}, & r = 3. \end{cases}$$

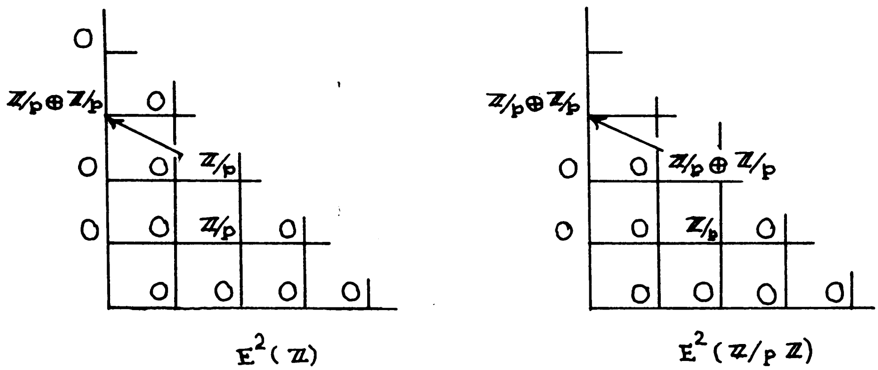
4. Calculations for the split extension. For the ring of dual numbers $\mathbf{F}[\varepsilon]$, we may argue very much as above. Define

$$\overline{\mathrm{SL}}(n, \mathbf{F}[\varepsilon]) = \mathrm{Ker}\left\{\mathrm{GL}(n, \mathbf{F}[\varepsilon]) \xrightarrow{\det(\ )^{p-1}} U(\mathbf{F}[\varepsilon])\right\}$$

and

$$\overline{\mathrm{SL}}(\mathbf{F}[\varepsilon]) = \varinjlim \overline{\mathrm{SL}}(n, \mathbf{F}[\varepsilon]).$$

The latter group is related to K-theory as before. Its prime to p homology is given by Quillen's results. To calculate $H_r(\overline{\mathrm{SL}}(n, \mathbf{F}[\varepsilon]), \mathbf{Z}; p)$, we argue as above except that in this case the extension $0 \to V_n \to \mathrm{SL}(n, \mathbf{F}[\varepsilon]) \to \mathrm{GL}(n, p) \to 1$ *splits*. The E^2-terms of the spectral sequences are as before.



To calculate the relevant d^2's, we rely on the following result from Chapter III.

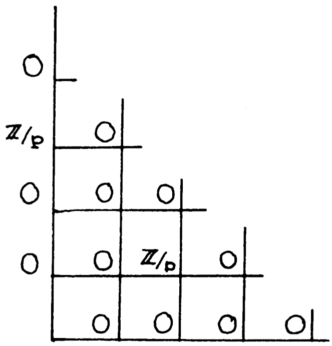
PROPOSITIONS 12.4(a), 12.3(b). *For $n = 2$, $p > 3$, $\mathrm{Im}\, d^2_{2,2}$ is 1 dimensional both for $\mathbf{Z}$ and $\mathbf{Z}/p\mathbf{Z}$.*

To determine $d^2_{2,2}$ for $n > 2$, consider the commutative diagrams

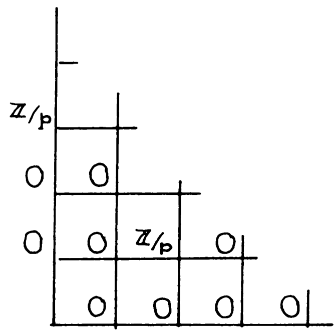
$$\begin{array}{ccc} H_0(\mathrm{GL}(n,p), H_3(V_n, C)) & \overset{d^2_{2,2}}{\leftarrow} & H_2(\mathrm{GL}(n,p), H_2(V_n, C)) \\ \uparrow\cong & & \uparrow\cong \\ H_0(\mathrm{GL}(2,p), H_3(V_2, C)) & \overset{d^2_{2,2}}{\leftarrow} & H_2(\mathrm{GL}(2,p), H_2(V_2, C)) \end{array}$$

for $C = \mathbf{Z}$ or $\mathbf{Z}/p\mathbf{Z}$. Note that as a result of our previous analysis, we know *both* vertical arrows are isomorphisms. Hence, dim Im $d^2_{2,2} = 1$ for arbitrary $n > 2$.

Thus the relevant E^3 terms look (stably) like:



$E^3(\mathbb{Z})$



$E^3(\mathbb{Z}/p\mathbb{Z})$

Hence, we have proved

THEOREM 4.1. *For $n \geqslant 2$, $p \geqslant 5$, we have stably*

$$\begin{aligned} H_1(\overline{\mathrm{SL}}(n, \mathbf{F}[\varepsilon]), \mathbf{Z}; p) &= 0, \\ H_2(\overline{\mathrm{SL}}(n, \mathbf{F}[\varepsilon]), \mathbf{Z}; p) &= 0, \\ H_3(\overline{\mathrm{SL}}(n, \mathbf{F}[\varepsilon]), \mathbf{Z}; p) &= \mathbf{Z}/p \oplus \mathbf{Z}/p, \\ H_4(\overline{\mathrm{SL}}(n, \mathbf{F}[\varepsilon]), \mathbf{Z}; p) &= 0. \end{aligned}$$

CHAPTER II. HOMOLOGY CALCULATIONS TO FILL IN THE BASIC TABLE

5. Initial calculations.

PROPOSITION 5.1. *For $p > 2$, $n \geqslant 1$, $1 \leqslant r \leqslant 2p - 4$,*

$$H_r(\mathrm{GL}(n,p), \mathbf{Z}; p) = 0, \qquad H_r(\mathrm{GL}(n,p), \mathbf{Z}/p\mathbf{Z}) = 0.$$

PROOF. Using the universal coefficient theorem and $B_n \supset U_n$ (a p-Sylow subgroup), we need only prove $H_r(B_n, \mathbf{Z}/p\mathbf{Z}) = 0$, $1 \leqslant r \leqslant 2p - 4$.

To establish this, note first that since $B_n = H_n \ltimes U_n$, we have

$$H_r(B_n, \mathbf{Z}/p\mathbf{Z}) = H_r(U_n, \mathbf{Z}/p\mathbf{Z})_{H_n}.$$

We look for the irreducible constituents (all one dimensional) of $H_r(U_n, \mathbf{Z}/p)$ as a $\mathbf{Z}/p(H_n)$-module. We shall see that, for the relevant r, these are all nontrivial–from which $H_r(U_n, \mathbf{Z}/p\mathbf{Z})_{H_n} = 0$ follows. $U = U_n$ may be built up as successive extensions

$$1 \to \frac{\Gamma_k}{\Gamma_{k+1}} \to \frac{U}{\Gamma_{k+1}} \to \frac{U}{\Gamma_k} \to 1, \qquad k = 2, 3, \dots, n-1,$$

where Γ_k is generated by $1 + e_{ij}, j - i \geqslant k$.

$$\Gamma_k \begin{bmatrix} 1 & 0 & * & * \\ & \ddots & \ddots & * \\ & & \ddots & 0 \\ & & & 1 \end{bmatrix}.$$

The E_2-term of the spectral sequence of this central extension is $H_*(U/\Gamma_k, \mathbf{Z}/p\mathbf{Z}) \otimes H_*(\Gamma_k/\Gamma_{k+1}, \mathbf{Z}/p)$ so we need only look at the irreducible H_n-constituents of this, or by induction the irreducible constituents of $H_*(U/\Gamma_2) \otimes H_*(\Gamma_2/\Gamma_3) \otimes \cdots \otimes H_*(\Gamma_{n-1})$. Recalling that this is a graded tensor product, we may identify this as the homology of the elementary abelian group

$$L_n = \bigoplus_k \Gamma_k/\Gamma_{k+1} = \bigoplus_{j-i \geqslant 1} \mathbf{Z}/p\mathbf{Z}e_{ij}$$

(upper triangular matrices with 0's on the diagonal).

As usual,

$$H_*(L_n, \mathbf{Z}/p\mathbf{Z}) \cong \underset{\text{degree 1}}{\underset{\uparrow}{\bigwedge L_n}} \otimes \underset{\text{degree 2}}{\underset{\uparrow}{\mathcal{P}(\rho L_n)}} \leftarrow \text{Divided power algebra}$$

Given the action, $h \cdot e_{ij} = t_i t_j^{-1} e_{ij}$ where

$$h = \begin{bmatrix} t_1 & & & 0 \\ & t_2 & & \\ & & \ddots & \\ 0 & & & t_n \end{bmatrix},$$

it is straightforward to see that the first *trivial* irreducible constituent in a positive degree occurs in degree $2(p-2) + 1 = 2p - 3$. (This is easier to see in the dual cohomology module which is an exterior algebra tensored with a symmetric algebra.) □

In what follows, we shall make frequent use of *the spectral sequence of a filtered module*. The general theory is as follows.

Let G be a group and A a G-module. Suppose in addition A is filtered by submodules $A = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots \supseteq A_l \supseteq A_{l+1} = 0$. We obtain from this a filtration of the standard chain complex

$$C_r(G, A) = F_0 C_r \supset F_{-1} C_r \supseteq F_{-2} C_r \supseteq \cdots \supseteq F_{-l} C_r \supseteq F_{-l-1} C_r = 0$$

with $F_sC_r = C_r(G, A_{-s})$ viewed naturally as a submodule of $C_r(G, A)$. In general, a filtered complex yields a spectral sequence. (See **[C-E**, Chapter XV] for an extensive discussion.) The E_1-term is given by

$$E^1_{s,t} = H_{s+t}\left(\frac{F_sC_*}{F_{s-1}C_*}\right) \cong H_{s+t}\left(C_*\left(G, \frac{A_{-s}}{A_{-s+1}}\right)\right)$$
$$= H_{s+t}\left(G, \frac{A_{-s}}{A_{-s+1}}\right).$$

Note that $E^1_{s,t} = 0$ *outside* the set defined by $s + t \geqslant 0$, $-l \leqslant s \leqslant 0$. To calculate $d^k_{s,t}(c^k)$ for $c^k \in E^k_{s,t}$, choose $\bar{c} \in F_sC_{s+t} = C_{s+t}(G, A_{-s})$ representing c^k such that $d\bar{c} \in F_{s-1}C_{s+t}$. (That is possible because $d^1c^1 = d^2c^2 = \cdots = d^{k-1}c^{k-1} = 0$ where $\bar{c}$ represents c^i in $E^i_{s,t}$.) Then $d^k(c^k)$ is represented by $d\bar{c}$.

PROPOSITION 5.2. *For $p \geqslant 5$, $n \geqslant 2$,*

$$H_0(\mathrm{GL}(n, p), V_n) = 0, \qquad H_0(\mathrm{GL}(n, p), M_n) = \mathbf{Z}/p\mathbf{Z}.$$
$$H_1(\mathrm{GL}(n, p), V_n) = H_1(\mathrm{GL}(n, p), M_n) = 0.$$
$$H_2(\mathrm{GL}(n, p), V_n) = H_2(\mathrm{GL}(n, p), M_n) = \mathbf{Z}/p\mathbf{Z}.$$

Moreover, this isomorphism is stable; *it commutes with the induced map from $n = 2$. $H_3(\mathrm{GL}(n, p), V_n) = H_3(\mathrm{GL}(n, p), M_n) = 0$. Similar formulas hold for B_n.*

PROOF. The sequence $0 \to V_n \to M_n \xrightarrow{\mathrm{Tr}} \mathbf{F} = \mathbf{Z}/p\mathbf{Z} \to 0$ together with Proposition 5.1 allows us to relate the homology of V_n to that of M_n or vice versa.

For V_n, we first calculate $H_r(B_n, V_n)$ $(r = 0, 1, 2, 3)$. We use $B_n = B_{n-1} \ltimes C_n$, $C_n = F^* \ltimes W_{n-1}$. Recall $[W, V]$ is the submodule of V spanned by all $u(s) - s$, $u \in W$, $s \in V$. Consider the tower

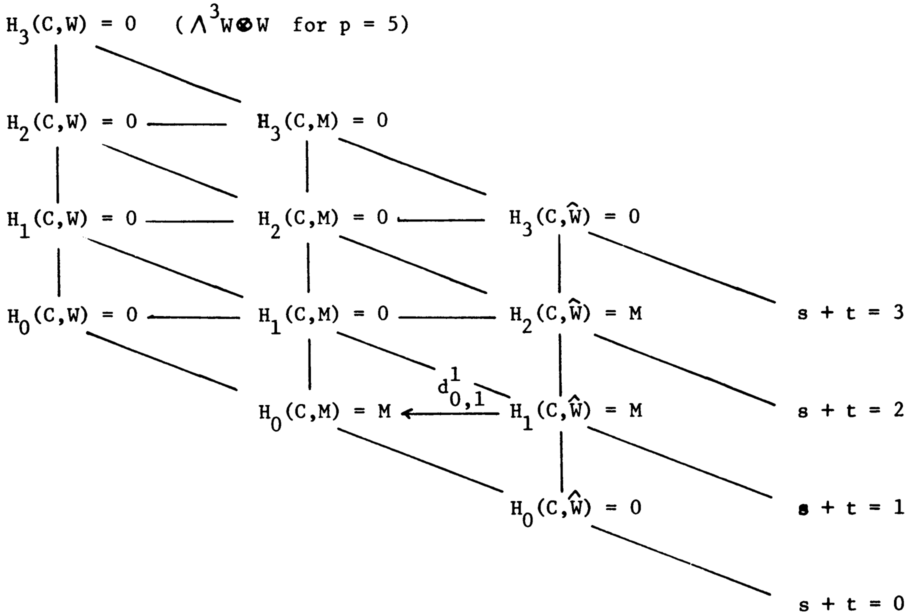
$$\begin{array}{lll} A_0 = V_n & \left.\right\} & \hat{W}_{n-1} \\ \quad | & & \\ A_1 = X_n = [W_{n-1}, V_n] & \left.\right\} & M_{n-1} \\ \quad | & & \\ A_2 = W_{n-1} = [W_{n-1}, X_n] & \left.\right\} & W_{n-1} \\ \quad | & & \\ \quad 0 & & \end{array}$$

where W_{n-1} acts trivially on the factors, $\mathbf{F}^*$ acts trivially on M_{n-1}, by multiplication on $\hat{W}_{n-1}$, and by reciprocal multiplication on W_{n-1}. Note $H_0(C_n, V_n) = H_0(W_{n-1}, V_n)_{\mathbf{F}^*} = (\hat{W}_{n-1})_{\mathbf{F}^*} = 0$.

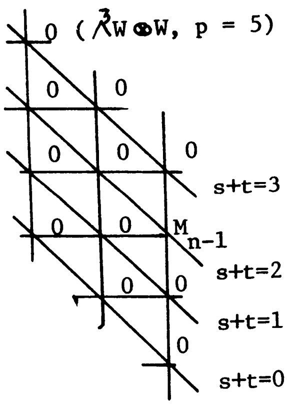
The given filtration yields a spectral sequence

$$E^1_{s,t} = H_{s+t}(C_n, A_{-s}/A_{-s+1}) \Rightarrow H_{s+t}(C_n, V_n).$$

The E^1-term is indicated below. (Subscripts n, $n - 1$ are deleted for convenience.)



(For example, $H_2(C, \hat{W}) = H_2(W, \hat{W})_{\mathbf{F}^*} = (\wedge^2 W \otimes \hat{W} \oplus W \otimes \hat{W})_{\mathbf{F}^*} = M$.) Since $H_0(C, V) = 0$, $d^1_{0,1}$ must be onto and hence it is an isomorphism. Hence, E^2 looks like:



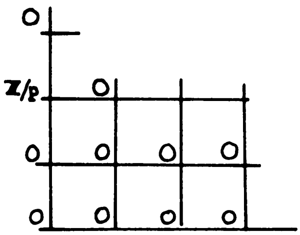
Thus, we have

LEMMA 5.3.

$$H_3(C_n, V_n) = 0\ (p > 5), \qquad H_2(C_n, V_n) \cong M_{n-1},$$
$$H_1(C_n, V_n) = 0, \qquad H_0(C_n, V_n) = 0.$$

For $p = 5$, $H_3(C_n, V_n)$ is a quotient of $\wedge^3 W_{n-1} \otimes W_{n-1}$.

We next consider the spectral sequence associated with the decomposition $B_n = B_{n-1} \ltimes C_n$. From $H_0(C_n, V_n) = 0$, we conclude $H_0(B_n, V_n) = 0$ for all $n \geq 1$. Replacing n by $n-1$, we see $H_0(B_{n-1}, M_{n-1}) = \mathbf{Z}/p\mathbf{Z}$. $H_1(B_1, M_1) = H_1(\mathbf{F}^*, \mathbf{F}) = 0$, so we may assume inductively that $H_1(B_{n-1}, M_{n-1}) = 0$. It follows easily that $H_r(B_n, V_n) = 0$, $r = 1, 3$, and $H_2(B_n, V_n) = \mathbf{Z}/p\mathbf{Z}$. See the E^2 diagram below. (For $p = 5$, it suffices to note that $H_0(H_{n-1}, \wedge^3 W_{n-1} \otimes W_{n-1}) = 0$ for H_{n-1} the subgroup of diagonal matrices.)



To complete the proof, we need only show the existence of a stable nonzero class in $H_2(\mathrm{GL}(n, p), M_n)$. (For, we know by the above that $H_2(\mathrm{GL}(n, p), M_n) \cong H_2(\mathrm{GL}(n, p), V_n) \leq \mathbf{Z}/p\mathbf{Z}$.) To see this, we show that

$$H_2(\mathrm{GL}(2, p), M_2) \to H_2(\mathrm{GL}(n, p), M_n)$$

is a monomorphism with the left-hand side $\neq 0$. Dually, we can show instead $H^2(\mathrm{GL}(n, p), \hat{M}_n) \to H^2(\mathrm{GL}(2, p), \hat{M}_2) \neq 0$ is an epimorphism. However, $A \otimes B \to \mathrm{Tr}(AB)$ provides a $\mathrm{GL}(n, p)$-pairing of $M_n \otimes M_n \to \mathbf{F}$ which gives an isomorphism $M_n \cong \hat{M}_n$. Under this isomorphism, the dual of the inclusion i: $M_2 \to M_n$ is the map j: $M_n \to M_2$ which given $A \in M_n$ picks out the 2×2 block of A in the upper left corner. Note $j \circ i = \mathrm{id}$.

From the comparison,

$$\begin{array}{ccccccccc} 1 & \to & M_2 & \to & \mathrm{GL}(2, p^2) & \to & \mathrm{GL}(2, p) & \to & 1: \iota_2 \in H^2(\mathrm{GL}_2, M_2) \\ & & \downarrow i & & \downarrow & & \downarrow k & & \\ 1 & \to & M_n & \to & \mathrm{GL}(n, p^2) & \to & \mathrm{GL}(n, p) & \to & 1: \iota_n \in H^2(\mathrm{GL}_n, M_n) \end{array}$$

we conclude $i_*(\iota_2) = k^*(\iota_n) \in H^2(\mathrm{GL}(2, p), M_n)$. However, the map under consideration is $j_* \circ k^*$: $H^2(\mathrm{GL}(n, p), M_n) \to H^2(\mathrm{GL}(2, p), M_2)$, so

$$j_*(k^*(\iota_n)) = j_*\big(i_*(\iota_2)\big) = \iota_2.$$

To complete the proof, we show that ι_2 is nontrivial. If ι_2 were trivial, we could find an element z in $\mathrm{GL}(2, p^2)$ in the coset of $[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}]$ such that $z^p = 1$. But z can be written $z = 1 + E_{12} + pA$ where

$$E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A \in M_n(\mathbf{Z}/p\mathbf{Z}).$$

Hence,

$$(1 + E_{12} + pA)^p = 1 + p(E_{12} + pA) + \binom{p}{2}(E_{12} + pA)^2 + \cdots + (E_{12} + pA)^p.$$

It is easy to see that $\binom{p}{j}(E_{12} + pA)^j = 0$, $2 \leq j < p$. Also, $(E_{12} + pA)^p = 0$ for $p \geq 5$ since every term has at least one factor E_{12}^2 or two factors pA. Hence, $z^p = 1 + pE_{12} \neq 1$.

6. Calculation of $H_*(C_n, V_n \wedge V_n)$. In what follows, we shall calculate $H_r(B_n, V_n \circ V_n)$ for $r = 0, 1$, $H_r(B_n, V_n \wedge V_n)$ for $r = 0, 1, 2$, $H_0(B_n, V_n \otimes \wedge^2 V_n)$, $H_r(B_n, \wedge^3 V_n)$ for $r = 0, 1$, and $H_0(B_n, \wedge^4 V_n)$. Our basic method is to couple the spectral sequence of a filtered module with appropriate coinvariant (dually invariant) calculations. We proceed generally in the indicated order except where necessary to pursue inductive arguments. In particular, we calculate $H_r(C_n, V_n \wedge V_n)$ (to be used primarily for $H_r(B_n, V_n \wedge V_n)$) in this section.

PROPOSITION 6.1. (0) *For $n \geqslant 2, p \geqslant 5$, $H_0(C_n, V_n \wedge V_n) = 0$.*

(i) *For $n \geqslant 3, p \geqslant 5$, there is an exact sequence*

$$0 \to H_1(C_n, V_n \wedge V_n) \to \frac{\operatorname{Ker}\{M_{n-1} \otimes M_{n-1} \to M_{n-1}\}}{d^1\left(\wedge^2 W_{n-1} \otimes \wedge^2 \hat{W}_{n-1}\right)} \to M_{n-1} \wedge M_{n-1} \to 0$$

which splits if $n \not\equiv -1 \bmod p$.

(ii) *For $n \geqslant 3, p > 5$ or $n \geqslant 4, p = 5$,*

$$H_2(C_n, V_n \wedge V_n) \cong \frac{M_{n-1} \otimes M_{n-1}}{d^1\left(\otimes^2 W_{n-1} \otimes \wedge^2 \hat{W}_{n-1}\right)}.$$

For $n = 3$, $p = 5$, there is an exact sequence $0 \to E \to H_2(C_3, V_3 \wedge V_3) \to M_2 \otimes M_2/\operatorname{Im} d^1 \to 0$, where E is a quotient of $\wedge^2 W_2 \otimes \wedge^2 W_2 \cong \mathbf{Z}/5\mathbf{Z}$.

(A fuller explanation of the notation appears below.)

PROOF. (0) for $n = 2$ follows since $A \wedge B \mapsto [A, B]$ defines an isomorphism $V_2 \wedge V_2 \cong V_2$.

We consider the following filtration of $V_n \wedge V_n$.

$$\left.\begin{array}{l} A_0 = V_n \wedge V_n \\ \quad | \\ A_1 = V_n \wedge X_n \end{array}\right\} \qquad \hat{W}_{n-1} \wedge \hat{W}_{n-1}$$

$$\left.\begin{array}{l} | \\ A_2 \\ | \end{array}\right\} \text{———} \left[\begin{array}{c} \left.\begin{array}{c} \hat{W}_{n-1} \otimes X_n \\ | \\ \hat{W}_{n-1} \otimes W_{n-1} \end{array}\right\} \hat{W}_{n-1} \otimes M_{n-1} \\ | \\ 0 \end{array}\right.$$

$$\left.\begin{array}{l} A_3 = X_n \wedge X_n \\ \quad | \\ A_4 = X_n \wedge W_{n-1} \end{array}\right\} \qquad M_{n-1} \wedge M_{n-1}$$

$$\left.\begin{array}{l} \quad | \\ A_5 = W_{n-1} \wedge W_{n-1} \end{array}\right\} \qquad M_{n-1} \otimes W_{n-1}$$

$$\begin{array}{l} | \\ 0 \end{array}$$

This induces the spectral sequence

$$E^1_{s,t} = H_{s+t}\left(C_n, \frac{A_{-s}}{A_{-s+1}}\right) \Rightarrow H_{s+t}(C_n, V_n \wedge V_n).$$

We exhibit the E^1-term for $p > 5$. (Subscripts and C_n are omitted for brevity.) (See diagram.) The necessary calculations and the case $p = 5$ are outlined in the appendix. The differentials are given as follows: we have the complementary inclusion onto a direct summand ($p > 2$)

$$i\colon \wedge^2 W \to W \otimes W, \qquad i(u \wedge v) = u \otimes v - v \otimes u.$$

Use i' for the corresponding morphism for $\hat{W}$. Then $d^1_{0,2}$ is the composition

$$\wedge^2 W \otimes \wedge^2 \hat{W} \overset{i \otimes i'}{\to} W \otimes W \otimes \hat{W} \otimes \hat{W} \overset{\text{twist}}{\underset{\sim}{\to}} W \otimes \hat{W} \otimes W \otimes \hat{W} = M \otimes M.$$

$d^1_{0,3}$ is the composition

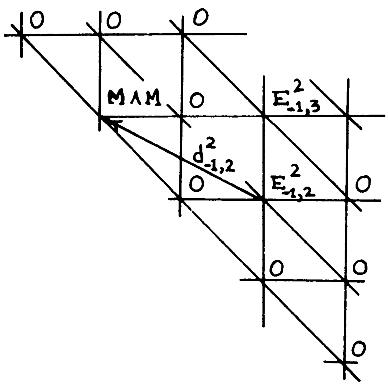
$$\otimes^2 W \otimes \wedge^2 \hat{W} \overset{-(\text{twist}) \otimes i'}{\to} \otimes^2 W \otimes \otimes^2 \hat{W} \overset{\text{twist}}{\underset{\sim}{\to}} M \otimes M.$$

Note that $d^1_{0,2}$ and $d^1_{0,3}$ are monomorphisms onto direct summands.

$d^2_{-1,2}\colon M \otimes M \to M$ is given by

$$d^1_{-1,2}(S \otimes T) = (T + \mathrm{Tr}(T))S,$$

which is easily seen to be an epimorphism. It follows that E^2 looks like

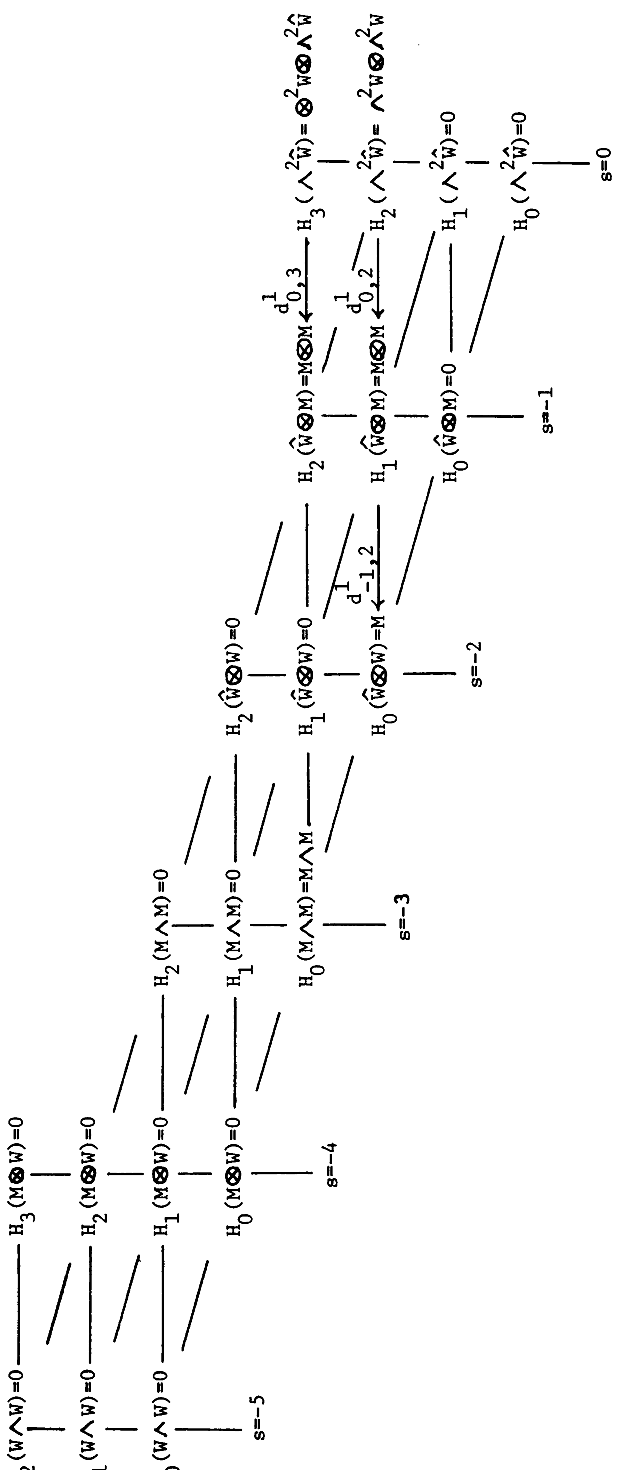


where $E^2_{-1,3} = \mathrm{Coker}(d^1_{0,3}\colon \otimes^2 W \otimes \wedge^2 \hat{W} \to W \otimes M)$ and

$$E^2_{-1,2} = \frac{\mathrm{Ker}\left(d^1_{-1,2}\colon M \otimes M \to M\right)}{\mathrm{Im}\left(d^1_{0,2}\colon \wedge^2 W \otimes \wedge^2 \hat{W} \to M \otimes M\right)}.$$

Then $d^2_{-1,2}$ is the *negative* of the morphism obtained by restricting the standard projection $\pi\colon M \otimes M \to M \wedge M$ to the kernel indicated above. (One sees easily that $\pi\colon M \otimes M \to M \wedge M$ kills $\mathrm{Im}\, d^1_{0,2}$.)

LEMMA 6.2. *$d^2_{-1,2}$ is an epimorphism; it is a split epimorphism if $n \not\equiv -1 \bmod p$.*



H_2(W∧W)=0
H_1(W∧W)=0
H_0(W∧W)=0
s=-5
H_3(M⊗W)=0
H_2(M⊗W)=0
H_1(M⊗W)=0
H_0(M⊗W)=0
s=-4
H_2(M∧M)=0
H_1(M∧M)=0
H_0(M∧M)=M∧M
s=-3
H_2(Ŵ⊗W)=0
H_1(Ŵ⊗W)=0
H_0(Ŵ⊗W)=M
d^1_{-1,2}
s=-2
H_2(Ŵ⊗M)=M⊗M
H_1(Ŵ⊗M)=M⊗M
H_0(Ŵ⊗M)=0
d^1_{0,3}
d^1_{0,2}
s=-1
H_3(∧^2Ŵ)=⊗^2W⊗∧^2Ŵ
H_2(∧^2Ŵ)=∧^2W⊗∧^2W
H_1(∧^2Ŵ)=0
H_0(∧^2Ŵ)=0
s=0

DIAGRAM

PROOF. It suffices to show $\mathrm{Ker}(M \otimes M \to M) \to M \wedge M$ is an epimorphism. $e_{ij} \wedge e_{kl} (i \neq k \text{ or } j \neq l)$ is the image of the following element in the kernel:

$$\begin{array}{ll} e_{ij} \otimes e_{kl} & \text{if } i \neq l \text{ and } k \neq l, \\ -e_{ki} \otimes e_{ij} & \text{if } i = l, \text{ but } i \neq j \text{ and } k \neq j, \\ \frac{1}{2}(e_{ii} \otimes e_{ki} - e_{ki} \otimes e_{ii}) & \text{if } i = j = l \text{ (then } k \neq i = j), \\ e_{ij} \otimes e_{ji} - \frac{1}{2} e_{jj} \otimes e_{jj} & \text{if } i = l, k = j \text{ (then } i \neq k = j), \\ -e_{kk} \otimes e_{ij} & \text{if } k = l \text{ but } j \neq k, i \neq j, \\ -\frac{1}{2}(e_{jj} \otimes e_{ij} - e_{ij} \otimes e_{jj}) & \text{if } k = l = j \text{ (then } i \neq k = j), \\ e_{ii} \otimes e_{kk} - \frac{1}{2}(e_{ii} \otimes e_{ii}) & \text{if } k = l \text{ and } i = j \text{ (then } j = i \neq k). \end{array}$$

For the splitting, define the $\mathrm{GL}(n-1, p)$-map $\psi \colon M \wedge M \to M \otimes M$ by

$$\psi(S \wedge T) = S \otimes T - T \otimes S + I \otimes [S, T] + I \otimes ((\mathrm{Tr}\, S)T - (\mathrm{Tr}\, T)S).$$

Note that $d^1_{-1,2}(\psi(S \wedge T)) = TS + (\mathrm{Tr}\, T)S - ST - (\mathrm{Tr}\, S)T + [S, T] + 0 \cdot I + ((\mathrm{Tr}\, S)T - (\mathrm{Tr}\, T)S)I + 0 \cdot I = 0$. Read modulo $\mathrm{Im}\, d^1_{0,2}$, ψ splits π since

$$\pi(\psi(S \wedge T)) = 2S \wedge T + I \wedge [S, T] + I \wedge (\mathrm{Tr}\, ST - \mathrm{Tr}\, TS).$$

[To see $\pi\psi$ is an isomorphism, examine its effect on the sequence $0 \to \bar{V} \to^{\lambda} M \wedge M \to \bar{V} \wedge \bar{V} \to 0$ $(\bar{V} = M/\mathbf{F}I)$ where $\lambda(\bar{S}) = I \wedge S$ (S representing $\bar{S}$). On $\lambda(\bar{V}) = I \wedge V$, $\pi\psi(I \wedge T) = 2I \wedge T + I \wedge (n-1)T = (n+1)I \wedge T$. $\pi\psi$ clearly induces multiplication by 2 modulo $I \wedge V$. Hence, for $p > 2$, $\pi\psi$ is an isomorphism $\Leftrightarrow n + 1 \not\equiv 0 \bmod p$.]

Looking at E^3, we are done.

Appendix. We indicate briefly how some of the terms in E^1 are calculated.

$$\begin{aligned} H_0(C_n, \wedge^2 \hat{W}) &= H_0(W, \wedge^2 \hat{W})_{\mathbf{F}^*} = (\wedge^2 \hat{W})_{\mathbf{F}^*} = 0, \\ H_1(C_n, \wedge^2 \hat{W}) &= (W \otimes \wedge^2 \hat{W})_{\mathbf{F}^*} = 0, \\ H_2(C_n, \wedge^2 \hat{W}) &= [(\wedge^2 W \oplus W) \otimes \wedge^2 \hat{W}]_{\mathbf{F}^*} = \wedge^2 W \otimes \wedge^2 \hat{W}, \\ H_3(C_n, \wedge^2 \hat{W}) &= [(\wedge^3 W \oplus \otimes^2 W) \otimes \wedge^2 \hat{W}]_{\mathbf{F}^*} = \otimes^2 W \otimes \wedge^2 \hat{W}, \\ H_1(C_n, \hat{W} \otimes M) &= (W \otimes \hat{W} \otimes M)_{\mathbf{F}^*} \cong M \otimes M, \\ H_2(C_n, \hat{W} \otimes M) &= [(\wedge^2 W \oplus W) \otimes \hat{W} \otimes M]_{\mathbf{F}^*} = W \otimes \hat{W} \otimes M = M \otimes M. \end{aligned}$$

The other calculations are similar. For $p = 5$ we have exceptionally

$$H_2(C_n, W \wedge W) = [(\wedge^2 W \oplus W) \otimes W \wedge W]_{\mathbf{F}^*} = \wedge^2 W \otimes \wedge^2 W$$

and $H_3(C_n, M \otimes W) = \wedge^3 W \otimes M \otimes W$ ($t^4 = 1$ in $\mathbf{F}_5^*$). Note for $n = 3$ $(n - 1 = 2)$, the latter term is zero, but $H_2(C_3, W_2 \wedge W_2) \cong \wedge^2 W_2 \otimes \wedge^2 W_2 \cong \mathbf{Z}/5\mathbf{Z}$.

To calculate the differentials we follow the procedure indicated in §5. See the table of notations for terminology.

$d^1_{0,2}$: $\wedge^2 W \subset H^2(W, \mathbf{Z}/p\mathbf{Z})$ is spanned by products $u \cap v$, so we need only calculate $d^1_{0,2}$ on elements of the form $u \cap v \otimes f \wedge g, f, g \in \hat{W}$. Choose

$$\bar{f} = \left[\begin{array}{c|c} 0 & 0 \\ \hline f & 0 \end{array}\right] \in V$$

to represent $f \in \hat{W}$ and similarly for g. We use the fact that d is a derivation relative to the Pontryagin product on the chain level and related facts outlined in §10.3. We have

$$\begin{aligned} d([u] \cap [v] \otimes \bar{f} \wedge \bar{g}) &= (d[u] \cap v - [u] \cap d[v]) \otimes \bar{f} \wedge \bar{g} \\ &= [(u[\cdot] - [\cdot]) \cap [v] - [u] \cap (v[\cdot] - [\cdot])] \otimes \bar{f} \wedge \bar{g} \\ &= [v] \otimes [(-u)(\bar{f} \wedge \bar{g}) - \bar{f} \wedge \bar{g}] \\ &\quad - [u] \otimes [(-v)(\bar{f} \wedge \bar{g}) - \bar{f} \wedge \bar{g}]. \end{aligned}$$

However, calculation shows

$$(-u)\bar{f} = \left[\begin{array}{c|c} -uf & -(fu)u \\ \hline f & fu \end{array}\right] \qquad (\text{note } fu = \operatorname{Tr}(uf))$$

and similarly for g. It is sufficient to calculate $(-u)(\bar{f} \wedge \bar{g}) - \bar{f} \wedge \bar{g}$ modulo A_2. That means we may ignore terms of the form

$$\left[\begin{array}{c|c} 0 & 0 \\ \hline * & 0 \end{array}\right] \wedge \left[\begin{array}{c|c} 0 & * \\ \hline 0 & 0 \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right] \wedge \left[\begin{array}{c|c} * & * \\ \hline 0 & * \end{array}\right].$$

We are left with $(-u)(\bar{f} \wedge \bar{g}) - \bar{f} \wedge \bar{g} \equiv \bar{f} \wedge \overline{(-ug)} + \overline{(-uf)} \wedge \bar{g}$ where

$$\bar{T} = \left[\begin{array}{c|c} T & 0 \\ \hline 0 & -\operatorname{Tr} T \end{array}\right].$$

Hence,

$$\begin{aligned} d([u] \cap [v] \otimes \bar{f} \wedge \bar{g}) &\equiv [v] \otimes [\bar{g} \wedge (\overline{uf}) - \bar{f} \wedge (\overline{ug})] \\ &\quad - [u] \otimes [\bar{g} \wedge (\overline{vf}) - \bar{f} \wedge (\overline{vg})]. \end{aligned}$$

In the identification $M \cong W \otimes \hat{W}$, $uf \cong u \otimes f$. Unraveling what the above expression represents in $H_1(C_n, \hat{W} \otimes M) = W \otimes \hat{W} \otimes M = W \otimes \hat{W} \otimes W \otimes \hat{W}$, we have

$$\begin{aligned} d^1_{0,2}(u \wedge v \otimes f \wedge g) &= (v \otimes g) \otimes (u \otimes f) - (v \otimes f) \otimes (u \otimes g) \\ &\quad - (u \otimes g) \otimes (v \otimes f) + (u \otimes f) \otimes (v \otimes g) \end{aligned}$$

which after twisting yields

$$(v \otimes u - u \otimes v) \otimes (g \otimes f - f \otimes g) = (u \otimes v - v \otimes u) \otimes (f \otimes g - g \otimes f)$$

as claimed.

$d^1_{0,3}$: $\otimes^2 W \subset H_3(W, \mathbf{Z}/p\mathbf{Z})$ is spanned by elements of the form $u \cap \rho(v)$ where $\rho(v) \in H_2(W, \mathbf{Z}/p\mathbf{Z})$ is represented by $\rho[v] = \sum_{i=0}^{p-1} [v, iv]$. We have

$$d([u] \cap \rho[v] \otimes \bar{f} \wedge \bar{g}) = (u[\cdot] - [\cdot]) \cap \rho[v] \otimes \bar{f} \wedge \bar{g} - [u] \cap d\rho[v] \otimes \bar{f} \wedge \bar{g}.$$

However,

$$d\rho[v] = d\sum [v, iv] = \sum (v[iv] - [(i+1)v] + [v])$$
$$= v\left(\sum [iv]\right) - \sum [iv].$$

Hence,

$$d([u] \cap \rho[v] \otimes \bar{f} \wedge \bar{g}) = \rho[v] \otimes \left[(-u)(\bar{f} \wedge \bar{g}) - \bar{f} \wedge \bar{g}\right]$$
$$- \sum [u] \cap [iv] \otimes \left[(-v)(\bar{f} \wedge \bar{g}) - \bar{f} \wedge \bar{g}\right].$$

Calculating modulo A_2 as above, we have

$$d([u] \cap \rho[v] \otimes \bar{f} \wedge \bar{g}) = \rho[v] \otimes \left[\bar{g} \wedge (\overline{uf}) - \bar{f} \wedge (\overline{ug})\right]$$
$$- \sum [u] \cap [iv] \otimes \left[\bar{g} \wedge (\overline{vf}) - \bar{f} \wedge (\overline{vg})\right].$$

Given $H_2(C_n, \hat{W} \otimes M) = [(\rho(W) \oplus \wedge^2 W) \otimes \hat{W} \otimes M]_{\mathbf{F}^*} \cong \rho(W) \otimes \hat{W} \otimes M \cong W \otimes \hat{W} \otimes M \cong M \otimes M$, and noting that the second group of terms is in $\wedge^2 W \otimes \hat{W} \otimes M$, we have

$$d^1_{0,3}(u \cap \rho(v) \otimes f \wedge g) = \rho(v) \otimes (g \otimes u \otimes f - f \otimes u \otimes g).$$

Identifying $\rho(V) \cong V$, we have

$$d^1_{0,3}(u \otimes v \otimes f \wedge g) = v \otimes g \otimes u \otimes f - v \otimes f \otimes u \otimes g$$
$$= -(v \otimes u) \otimes (f \otimes g - g \otimes f) \quad \text{(after twisting)}.$$

$d^1_{-1,2}$ and $d^2_{-1,2}$: $H_1(C_n, \hat{W} \otimes M) = W \otimes \hat{W} \otimes M \cong M \otimes M$. Represent $u \otimes f \otimes T$ by $[u] \otimes \bar{f} \wedge \bar{T}$. Then

$$d([u] \otimes \bar{f} \wedge \bar{T}) = (u[\cdot] - [\cdot]) \otimes \bar{f} \wedge \bar{T}$$
$$= [\cdot] \otimes \left[(-u)(\bar{f} \wedge \bar{T}) - \bar{f} \wedge \bar{T}\right].$$

However,

$$(-u)\bar{f} = \bar{f} - (\overline{uf}) - (fu)\bar{u} \quad \text{where } \bar{u} = \left[\begin{array}{c|c} 0 & u \\ \hline 0 & 0 \end{array}\right].$$

Also $(-u)\bar{T} = \bar{T} + \overline{T'u}$ where $T'u = (T + \operatorname{Tr} T)u$. Thus, modulo $A_4 = X \wedge W$,

$$(*) \qquad d([u] \otimes \bar{f} \wedge \bar{T}) \equiv [\cdot] \otimes \left[\bar{f} \wedge \overline{T'u} - \overline{uf} \wedge \bar{T}\right].$$

Hence, under the identification $u \otimes f \simeq uf \in M$, $\hat{W} \otimes W \cong W \otimes \hat{W} \cong M$, we have $d^1_{-1,2}(uf \otimes T) = (T'u)f = T'(uf)$. Since the matrices uf span M, we may write

$$d^1_{-1,2}(S \otimes T) = (T + \operatorname{Tr}(T))S$$

as claimed.

Examining formula $(*)$ a bit more carefully, we see that $d^2_{-1,2}$ is represented by the *restriction* to Ker $d^1_{1,2}$ of the morphism defined by $uf \otimes T \mapsto -uf \wedge T$ or $S \otimes T \mapsto -S \wedge T$ as claimed.

It remains to deal with the case $p = 5$. We show

$$d^1_{-4,7}\colon H_3(C_n, M \otimes W) = \wedge^3 W \otimes M \otimes W \to H_3(C_n, W \wedge W)$$
$$= \wedge^2 W \otimes \wedge^2 W$$

is an epimorphism. Then $E^2_{-5,6}=0$ and the argument proceeds as before. To calculate $d^1_{-4,7}$, we calculate

$$d([u]\cap([v]\cap[w])\otimes\overline{T}\wedge\bar{x}),\qquad u,v,w,x\in W, T\in M.$$

Use $(-u)\overline{T}=\overline{T}+\overline{T'u}$, $(-u)\bar{x}=\bar{x}$ and the fact that d is a derivation with $d[u]=u[\cdot]-[\cdot]$. The above expression is

$$[v]\cap[w]\otimes\overline{T'u}\wedge\bar{x}-[u]\cap[w]\otimes\overline{T'v}\wedge\bar{x}+[u]\cap[v]\otimes\overline{T'w}\otimes\bar{x}.$$

Hence,

$$\begin{aligned}&d^1_{-4,7}(u\wedge v\wedge w\otimes T\otimes x)\\&\qquad=v\wedge w\otimes T'u\wedge x-u\wedge w\otimes T'v\wedge x+u\wedge v\otimes T'w\wedge x\end{aligned}$$

where $T'=T+\mathrm{Tr}(T)$.

To see this is an epimorphism, note that, for $l\neq i,j,k$, $e_i\wedge e_j\wedge e_l\otimes e_{kl}\otimes x\mapsto e_i\wedge e_j\otimes e_k\wedge x$. For $n=4$ $(n-1=3)$

$$e_1\wedge e_2\wedge e_3\otimes e_{13}\otimes x\mapsto e_1\wedge e_2\otimes e_1\wedge x.$$

Similarly, $e_1\wedge e_2\otimes e_2\wedge x$, $e_2\wedge e_3\otimes e_2\wedge x$, $e_2\wedge e_3\otimes e_3\wedge x$, $e_1\wedge e_3\otimes e_1\wedge x$, and $e_1\wedge e_3\otimes e_3\wedge x$ are in the image. Since x is arbitrary, we are done with the case $n\geqslant 4$. For $n=3$, there is an additional nonzero term $\bigwedge^2 W_2\otimes\bigwedge^2 W_2\cong\mathbf{Z}/5\mathbf{Z}$ (at the upper left corner of the diagram) in E^2. It may persist in E^∞ leading to the exact sequence given in the statement of the proposition. □

7. Homology of $V_n\circ V_n$ and $V_n\wedge V_n$.

PROPOSITION 7.1. *For $n\geqslant 2$,*

$$H_0(B_n,V_n\circ V_n)=H_0(\mathrm{GL}(n,p),V_n\circ V_n)=\mathbf{Z}/p\mathbf{Z}\,\overline{e_{12}\circ e_{21}},$$

and

$$H_0(B_n,M_n\circ M_n)=H_0(\mathrm{GL}(n,p),M_n\circ M_n)=\mathbf{Z}/p\mathbf{Z}\,\overline{e_{11}\circ e_{11}}+\mathbf{Z}/p\mathbf{Z}\,\overline{e_{12}\circ e_{21}}.$$

PROOF. We start with $M_n\circ M_n$. Write $[x,a]=x(a)-a$ for $x\in B_n$, $a\in M_n\circ M_n$. Using the action of diagonal matrices, we see that $e_{ij}\circ e_{kl}\equiv 0 \bmod [B_n, M_n\circ M_n]$ unless $i\neq j$, $k=j$, $l=i$ or $i=j$, $k=l$. Hence, $H_0(B_n,M_n\circ M_n)$ is spanned by the cosets of $e_{ij}\circ e_{ji}$, $i<j$, and $e_{ii}\circ e_{jj}$, $i\leqslant j$. Consideration of $[x_{si},e_{ij}\circ e_{js}]$ shows $e_{sj}\circ e_{js}\equiv e_{ij}\circ e_{ji}$ for $s<i<j$. Similarly, $e_{it}\circ e_{ti}\equiv e_{ij}\circ e_{ji}$ for $i<j<t$. Consideration of $[x_{si},e_{is}\circ e_{jj}]$ shows $e_{ss}\circ e_{jj}\equiv e_{ii}\circ e_{jj}$ for $s<i<j$, and similarly $e_{ii}\circ e_{jj}\equiv e_{ii}\circ e_{tt}$ for $i<j<t$. Finally, consideration of $[x_{ij},e_{ji}\circ e_{ii}]$ shows $e_{ii}\circ e_{jj}\equiv e_{ii}\circ e_{ii}-e_{ij}\circ e_{ji}$ for $i<j$. It follows that $e_{11}\circ e_{11}$ and $e_{12}\circ e_{21}$ span. That these are independent follows by considering the values assumed by the B_n-invariant forms defined by $\phi(A\circ B)=\mathrm{Tr}(A)\mathrm{Tr}(B)$ and $\psi(A\circ B)=\mathrm{Tr}(AB)$. In fact, since those functionals are $\mathrm{GL}(n,p)$-invariant, the result also follows for the larger group.

To obtain the result for $V_n\circ V_n$, consider the exact sequence $0\to V_n\circ V_n\to M_n\circ M_n\xrightarrow{\lambda^+}M_n\to 0$ where $\lambda^+(A\circ B)=(\mathrm{Tr}\,A)B+(\mathrm{Tr}\,B)A$. Note $\lambda^+(e_{12}\circ e_{21})=0$ and use Proposition 5.2. □

COROLLARY 7.2. *In* $H_0(B_n, M_n \circ M_n)$,

$$e_{11} \circ e_{22} \equiv e_{11} \circ e_{11} - e_{12} \circ e_{21}.$$

PROPOSITION 7.3. *For* $n \geq 2$,

$$H_0(B_n, V_n \wedge V_n) = 0, \qquad H_0(B_n, M_n \wedge M_n) = 0,$$
$$H_0(B_n, V_n \otimes V_n) = \mathbf{Z}/p\mathbf{Z}\,\overline{e_{12} \otimes e_{21}},$$
$$H_0(B_n, M_n \otimes M_n) = \mathbf{Z}/p\mathbf{Z}\,\overline{e_{11} \otimes e_{11}} + \mathbf{Z}/p\mathbf{Z}\,\overline{e_{12} \otimes e_{21}}.$$

Similarly for $\mathrm{GL}(n, p)$.

PROOF. Use Proposition 6.1(0) to derive the first statement. For the second, use the exact sequence

$$0 \to V_n \wedge V_n \to M_n \wedge M_n \xrightarrow{\lambda^-} V_n \to 0,$$

where $\lambda^-(A \wedge B) = (\mathrm{Tr}\,A)B - (\mathrm{Tr}\,B)A$, and then use Proposition 5.2.

The third result follows from Proposition 7.1 and the decomposition $V_n \otimes V_n = \wedge^2 V_n \oplus S^2 V_n$ ($p > 2$). The last result is analogous. □

We shall need some congruences in $H_0(B_n, M_n \otimes M_n)$ for later reference.

PROPOSITION 7.4. *For* $n \geq 2$, *we have*

$$e \otimes f \equiv f \otimes e, \qquad e_{11} \otimes e_{22} \equiv e_{11} \otimes e_{11} - e_{12} \otimes e_{21}$$

modulo $[B_n, M_n \otimes M_n]$.

PROOF. Use $H_0(B_n, \wedge^2 V_n) = 0$ and Corollary 7.2.

PROPOSITION 7.5. *For* $n \geq 2$, $H_1(B_n, V_n \circ V_n) = H_1(\mathrm{GL}(n, p), V_n \circ V_n) = 0$. *Also*, $H_1(B_n, M_n \circ M_n) = H_1(\mathrm{GL}(n, p), M_n \circ M_n) = 0$.

PROOF. Note first that for each n the first statement implies the second. This follows from Proposition 5.2 and the sequence $0 \to V_n \circ V_n \to M_n \circ M_n \xrightarrow{\lambda^+} M_n \to 0$.

We argue inductively. For $n = 2$, $H_1(B_2, V_2 \circ V_2) = H_1(U_2, V_2 \circ V_2)_{H_2}$. Let x generate the cyclic group U_2. As an H_2-module

$$H_1(U_2, V_2 \circ V_2) = W_1 \otimes \frac{(V_2 \circ V_2)^{U_2}}{T_x(V_2 \circ V_2)}$$

where $T_x = 1 + x + x^2 + \cdots + x^{p-1}$. Direct calculation shows $(V_2 \circ V_2)^{U_2} = \mathbf{Z}/p\mathbf{Z}(h_1 \circ h_1 + 4e_{21} \circ e_{12}) + \mathbf{Z}/p\mathbf{Z}e_{12} \circ e_{12}$.

$$t = \begin{bmatrix} t_1 & 0 \\ 0 & t_2 \end{bmatrix}$$

acts trivially on the first factor and as $(t_1 t_2^{-1})^2$ on the second. Since it acts as t_1 on W_1, $W_1 \otimes (V_2 \circ V_2)^{U_2}/T_x(V_2 \circ V_2)$ does not have a trivial irreducible H_2-constituent. Hence, $H_1(B_2, V_2 \circ V_2) = 0$ as required. For $n > 2$, we consider the following filtration.

$$
\left.\begin{array}{l} A_0 = \mathbb{S}^2 V_n \\ | \\ A_1 = V_n \circ X_n \end{array}\right\} \qquad \mathbb{S}^2 \hat{W}_{n-1} \cdots
$$

$$
\left.\begin{array}{l} | \\ A_2 \\ | \\ A_3 = X_n \circ X_n \end{array}\right\} \qquad \left\{\begin{array}{c} \hat{W}_{n-1} \otimes X_n \\ | \\ \hat{W}_{n-1} \otimes W_{n-1} \\ | \\ 0 \end{array}\right\} \hat{W}_{n-1} \otimes M_{n-1}
$$

$$
\left.\begin{array}{l} | \\ A_4 = X_n \circ W_{n-1} \end{array}\right\} \qquad \mathbb{S}^2 M_{n-1}
$$

$$
\left.\begin{array}{l} | \\ A_5 = W_{n-1} \circ W_{n-1} \end{array}\right\} \qquad M_{n-1} \otimes W_{n-1}
$$

$$
\begin{array}{l} | \\ 0 \end{array}
$$

This filtration yields a spectral sequence

$$
E^1_{s,t} = H_{s+t}\left(B_n, \frac{A_s}{A_{-s+1}} \right) \Rightarrow H_{s+t}(B_n, A).
$$

To calculate the E^1-term, we proceed as follows: Each $H_r(B_n, A_s/A_{-s+1})$ is calculated using the spectral sequence of the split extension $B_n = B_{n-1} \ltimes C_n$ ($C_n = \mathbf{F}^* \rtimes W_{n-1}$).

$s = 0$. $H_0(C_n, \mathbb{S}^2\hat{W}_{n-1}) = (\mathbb{S}^2\hat{W}_{n-1})_{\mathbf{F}^*} = 0$, so

$$
H_0\big(B_n, \mathbb{S}^2\hat{W}_{n-1}\big) = H_0\big(B_{n-1}, H_0\big(C_n, \mathbb{S}^2\hat{W}_{n-1}\big)\big) = 0.
$$

$H_1(C_n, \mathbb{S}^2\hat{W}_{n-1}) = (\hat{W}_{n-1} \otimes \mathbb{S}^2\hat{W}_{n-1})_{\mathbf{F}^*} = 0$, so

$$
H_1\big(B_{n-1}, \mathbb{S}^2\hat{W}_{n-1}\big) = H_0\big(B_{n-1}, H_1\big(C_n, \mathbb{S}^2\hat{W}_{n-1}\big)\big) = 0.
$$

$H_2(C_n, \mathbb{S}^2\hat{W}_{n-1}) = \wedge^2 W_{n-1} \otimes \mathbb{S}^2\hat{W}_{n-1}$, so

$$
H_2\big(B_n, \mathbb{S}^2\hat{W}_{n-1}\big) = H_0\big(B_{n-1}, H_2\big(C_n, \mathbb{S}^2\hat{W}_{n-1}\big)\big) = H_0\big(B_{n-1}, \wedge^2 W_{n-1} \otimes \mathbb{S}^2\hat{W}_{n-1}\big).
$$

By Proposition 7.3, $H_0(B_{n-1}, M_{n-1} \otimes M_{n-1}) = \mathbf{Z}/p\mathbf{Z}\overline{e_{11} \otimes e_{11}} + \mathbf{Z}/p\mathbf{Z}\overline{e_{12} \otimes e_{21}}$. Also, $\wedge^2 W \otimes \mathbb{S}^2\hat{W}$ is a direct summand of $(W \otimes W) \otimes (\hat{W} \otimes \hat{W}) \cong M \otimes M$. Hence, $H_0(B_{n-1}, \wedge^2 W_{n-1} \otimes \mathbb{S}^2\hat{W}_{n-1})$ is generated by the projections of $e_{11} \otimes e_{11}$ and $e_{12} \otimes e_{21}$. However,

$$
e_{11} \otimes e_{11} \simeq e_1 \otimes e_1 \otimes e^1 \otimes e^1 \mapsto e_1 \wedge e_1 \otimes e^1 \circ e^1 = 0,
$$

$$
e_{12} \otimes e_{21} \simeq e_1 \otimes e_2 \otimes e^2 \otimes e^1 \mapsto e_1 \wedge e_2 \otimes e^2 \circ e^1.
$$

Going the other way yields

$$
\begin{aligned}
e_1 \wedge e_2 \otimes e^2 \circ e^1 &\mapsto \tfrac{1}{2}(e_1 \otimes e_2 - e_2 \otimes e_1) \otimes \tfrac{1}{2}(e^2 \otimes e^1 + e^1 \otimes e^2) \\
&\simeq \tfrac{1}{4}[e_{12} \otimes e_{21} - e_{22} \otimes e_{11} + e_{11} \otimes e_{22} - e_{21} \otimes e_{12}].
\end{aligned}
$$

However, we know from Proposition 7.4 that $e_{12} \otimes e_{21} \equiv e_{21} \otimes e_{12}$, $e_{11} \otimes e_{22} \equiv e_{22} \otimes e_{11}$, so $e_1 \wedge e_2 \otimes e^2 \circ e^1 \mapsto 0$. Hence, $H_2(B_n, \mathbb{S}^2\hat{W}_{n-1}) = 0$.

$s = -1$. $H_0(C_n, \hat{W}_{n-1} \otimes M_{n-1}) = 0$, so $H_0(B_n, \hat{W}_{n-1} \otimes M_{n-1}) = 0$.

$H_1(C_n, \hat{W}_{n-1} \otimes M_{n-1}) = M_{n-1} \otimes M_{n-1}$, so $H_1(B_n, \hat{W}_{n-1} \otimes M_{n-1}) = H_0(B_{n-1}, M_{n-1} \otimes M_{n-1}) = \mathbf{Z}/pe_{11} \otimes e_{11} \oplus \mathbf{Z}/pe_{12} \otimes e_{21}$.

$s = -2$. $H_0(C_n, \hat{W}_{n-1} \otimes W_{n-1}) = M_{n-1}$, so $H_0(B_n, \hat{W}_{n-1} \otimes W_{n-1}) = H_0(B_{n-1}, M_{n-1}) = \mathbf{Z}/p\mathbf{Z}e_{11}$.

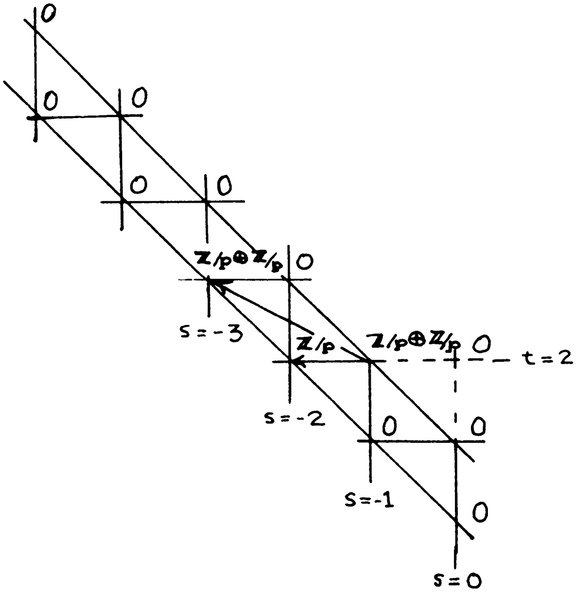
$H_1(C_n, \hat{W}_{n-1} \otimes W_{n-1}) = 0$, so $H_1(B_n, \hat{W}_{n-1} \otimes W_{n-1}) = H_1(B_{n-1}, M_{n-1}) = 0$.

$s = -3$. $H_0(C_n, \mathbb{S}^2M_{n-1}) = \mathbb{S}^2M_{n-1}$, so $H_0(B_n, \mathbb{S}^2M_{n-1}) = H_0(B_{n-1}, \mathbb{S}^2M_{n-1}) = \mathbf{Z}/p\mathbf{Z}e_{11} \otimes e_{11} + \mathbf{Z}/p\mathbf{Z}e_{12} \otimes e_{21}$.

$H_1(C_n, \mathbb{S}^2M_{n-1}) = 0$, so $H_1(B_n, \mathbb{S}^2M_{n-1}) = H_1(B_{n-1}, \mathbb{S}^2M_{n-1}) = 0$ (assumed inductively).

$s \leqslant -4$. $H_r(C_n, A_s/A_{-s+1}) = 0$, $r = 0, 1$, in each case. Hence, $H_r(B_n, A_s/A_{-s+1}) = 0$, $r = 0, 1$.

The $E^1_{s,t}$-term of the spectral sequence looks like



Since $H_0(B_n, V_n \circ V_n) = \mathbf{Z}/p\mathbf{Z}e_{12} \circ e_{21}$, it follows easily that $E^3_{-1,2} = 0$, so $H_1(B_n, V_n \circ V_n) = 0$. (In fact, since $e_{12} \circ e_{21}$ is in position $(-3, 3)$, we can say further that $d^1_{-1,2}$ is an epimorphism and $d^2_{-1,2}$ a monomorphism.)

Proposition 7.6. *For $n \geqslant 2$, $p \geqslant 5$, we have*

(a)

$$H_1(B_n, V_n \wedge V_n) = H_1(\mathrm{GL}(n, p), V_n \wedge V_n) = 0;$$
$$H_1(B_n, M_n \wedge M_n) = H_1(\mathrm{GL}(n, p), M_n \wedge M_n) = 0;$$

(b)

$$H_2(B_n, V_n \wedge V_n) = H_2(\mathrm{GL}(n, p), V_n \wedge V_n) = \mathbf{Z}/p\mathbf{Z}.$$

PROOF. First note that using Proposition 5.2 and the sequence $0 \to V_n \wedge V_n \to M_n \wedge M_n \to V_n \to 0$ we derive $H_1(B_n, M_n \wedge M_n) = 0$ from $H_1(B_n, V_n \wedge V_n) = 0$.

To prove (a) and (b) we argue inductively.

For $n = 2$, as in the proof of Proposition 6.1(0), $H_1(B_2, V_2 \wedge V_2) = H_1(B_2, V_2) = 0$, and $H_2(B_2, V_2 \wedge V_2) = H_2(B_2, V_2) = \mathbf{Z}/p\mathbf{Z}$ by Proposition 5.2.

For $n > 2$, use the spectral sequence of the group extension $B_n = B_{n-1} \ltimes C_n$. To prove (a), since $H_1(B_{n-1}, H_0(C_n, V_n \wedge V_n)) = 0$ by Proposition 6.1, we need only show $H_0(B_{n-1}, H_1(C_n, V_n \wedge V_n)) = 0$. From Proposition 7.3,

$$H_0(B_{n-1}, M_{n-1} \wedge M_{n-1}) = 0,$$

and we may assume inductively that $H_1(B_{n-1}, M_{n-1} \wedge M_{n-1}) = 0$. Hence, by Proposition 6.1,

$$H_0(B_{n-1}, H_1(C_n, V_n \wedge V_n)) \cong H_0(B_{n-1}, \mathrm{Ker}/\mathrm{Im})$$

which in turn is a direct summand of $H_0(B_{n-1}, \mathrm{Ker})$. Since $H_1(B_{n-1}, M_{n-1}) = 0$,

$$\begin{array}{ccccc} 0 \to H_0(B_{n-1}, \mathrm{Ker}) & \to & H_0(B_{n-1}, M_{n-1} \otimes M_{n-1}) & \to & H_0(B_{n-1}, M_{n-1}) \to 0 \\ & & \| & & \| \\ & & \mathbf{Z}/p\mathbf{Z}\,\overline{e_{12} \otimes e_{21}} + \mathbf{Z}/p\mathbf{Z}\,\overline{e_{11} \otimes e_{11}} & & \mathbf{Z}/p\mathbf{Z}\,\overline{e_{11}} \end{array}$$

is exact. Since $e_{12} \otimes e_{21} \mapsto e_{11}$ and $e_{11} \otimes e_{11} \mapsto 2e_{11}$,

$$H_0(B_{n-1}, \mathrm{Ker}) = \mathbf{Z}/p\mathbf{Z}\left(2\,\overline{e_{12} \otimes e_{12}} - \overline{e_{11} \otimes e_{11}}\right).$$

Consider next the morphism $d^1_{0,3}\colon \wedge^2 W \otimes \wedge^2 \hat{W} \to \otimes^2 W \otimes \otimes^2 W \cong M \otimes M$.

$$\begin{aligned} e_1 \wedge e_2 \otimes e^1 \wedge e^2 &\mapsto (e_1 \otimes e_2 - e_2 \otimes e_1) \otimes (e^1 \otimes e^2 - e^2 \otimes e^1) \\ &\equiv e_{11} \otimes e_{22} - e_{12} \otimes e_{21} - e_{21} \otimes e_{12} + e_{22} \otimes e_{11} \\ \text{(Proposition 7.4)} \qquad &\equiv 2(e_{11} \otimes e_{22} - e_{12} \otimes e_{21}) \\ &\equiv 2(e_{11} \otimes e_{11} - e_{12} \otimes e_{21} - e_{12} \otimes e_{21}) \\ &\equiv 2(e_{11} \otimes e_{11} - 2e_{12} \otimes e_{21}). \end{aligned}$$

Hence, $H_0(B_{n-1}, \mathrm{Ker}/\mathrm{Im}) = 0$ as required.

We next consider (b). By Proposition 6.1, $H_0(B_{n-1}, H_2(V_n \wedge V_n)) = H_0(B_{n-1}, (M_{n-1} \otimes M_{n-1})/\mathrm{Im}\, d^1)$ which (as shown in the proof of 6.1) is a direct summand of $H_0(B_{n-1}, M_{n-1} \otimes M_{n-1}) = \mathbf{Z}/p\mathbf{Z}\overline{e_{12} \otimes e_{21}} + \mathbf{Z}/p\mathbf{Z}\overline{e_{11} \otimes e_{11}}$. (For $n = 3$, $p = 5$, use the fact that $H_0(B_2, \wedge^2 W_2 \otimes \wedge^2 W_2) = 0$.) Referring again to the proof of 6.1, we recall that $\mathrm{Im}\, d^1 \cong \otimes^2 W_{n-1} \otimes \wedge^2 \hat{W}_{n-1}$. Note that $e_{11} \otimes e_{11} \simeq (e_1 \otimes e_1) \otimes (e^1 \otimes e^1) \mapsto e_1 \otimes e_1 \otimes e^1 \wedge e^1 = 0$, so $H_0(B_{n-1}, \mathrm{Im}\, d^1) \leqslant \mathbf{Z}/p\mathbf{Z}$. On the other hand, going the other way,

$$\begin{aligned} e_1 \otimes e_2 \otimes e^1 \wedge e^2 &\mapsto e_1 \otimes e_2 \otimes (e^1 \otimes e^2 - e^2 \otimes e^1) \\ &\cong e_{11} \otimes e_{22} - e_{12} \otimes e_{21} \not\equiv 0. \end{aligned}$$

Hence, $H_0(B_{n-1}, \operatorname{Im} d^1) = \mathbf{Z}/p\mathbf{Z}$, and thus

$$H_0\big(B_{n-1}, (M_{n-1} \otimes M_{n-1})/\operatorname{Im} d^1\big) = \mathbf{Z}/p\mathbf{Z}.$$

Continuing with (b), we shall show that $H_1(B_{n-1}, H_1(C_n, V_n \wedge V_n)) = 0$. We use the short exact sequence

$$0 \to H_1(C_n, V_n \wedge V_n) \to \frac{\operatorname{Ker}(M_{n-1} \otimes M_{n-1} \to M_{n-1})}{\operatorname{Im} d^1}$$

$$\stackrel{\pi}{\to} M_{n-1} \wedge M_{n-1} \to 0$$

of Proposition 6.1(i). If $n \not\equiv -1 \bmod p$, the sequence splits, so we are reduced to showing $H_1(B_{n-1}, \operatorname{Ker}/\operatorname{Im}) = 0$. If $n \equiv -1 \bmod p$, we obtain the same reduction by means of the following argument (suggested by the referee). It suffices to show that $H_2(B_{n-1}, \operatorname{Ker}) \to H_2(B_{n-1}, M_{n-1} \wedge M_{n-1})$ is an epimorphism. In the proof of 6.1, we derived the splitting (for $n \not\equiv -1$) by showing that $\pi\psi$ is an isomorphism of $M_{n-1} \wedge M_{n-1}$. (Refer to Lemma 6.2 for notation.) That in turn was deduced by examining the effect of $\pi\psi$ on the short exact sequence $0 \to \overline{V}_{n-1} \stackrel{\lambda}{\to} M_{n-1} \otimes M_{n-1} \stackrel{\chi}{\to} \overline{V}_{n-1} \wedge \overline{V}_{n-1} \to 0$ (where $\lambda(\overline{T}) = I_{n-1} \wedge T$). If $n \equiv -1$, $\pi\psi$ is unfortunately trivial on $\operatorname{Im} \lambda$. However, we still have $\chi\pi\psi = 2\chi$ so $\operatorname{Im}(2\operatorname{id} - \pi\psi) \subseteq \operatorname{Im} \lambda$. Thus, it suffices to show $\lambda_*(H_2(B_{n-1}, \overline{V}_{n-1})) \subseteq \pi_*(H_2(B_{n-1}, \operatorname{Ker}))$. For this, consider σ: $M_{n-2} \to M_{n-1} \otimes M_{n-1}$ defined by

$$\sigma(T) = I_{n-2} \otimes T - T \otimes e_{n-1,n-1} - (\operatorname{Tr}(T)/(n-1))I_{n-2} \otimes I_{n-2}.$$

Note that $\operatorname{Im} \sigma \subseteq \operatorname{Ker}(M_{n-1} \otimes M_{n-1} \to M_{n-1})$. Also, if we set $\iota(T) = \overline{T}$, then

$$\begin{array}{ccc} M_{n-2} & \stackrel{\sigma}{\to} & \operatorname{Ker}(M_{n-1} \otimes M_{n-1} \to M_{n-1}) \\ \downarrow \iota & & \downarrow \pi \\ \overline{V}_{n-1} & \stackrel{\lambda}{\to} & M_{n-1} \wedge M_{n-1} \end{array}$$

commutes. Since $\iota_*: H_2(B_{n-2}, M_{n-2}) \to H_2(B_{n-1}, \overline{V}_{n-1})$ is an isomorphism by Propositions 5.1 and 5.2, it follows readily that $\lambda_*(H_2(B_{n-1}, \overline{V}_{n-1})) \subseteq \operatorname{Im} \pi_*$ as required.

To show that $H_1(B_{n-1}, \operatorname{Ker}/\operatorname{Im}) = 0$, we use the fact established in the proof of Proposition 6.1 that Im is the direct summand of Ker; thus, we need only show that $H_1(B_{n-1}, \operatorname{Ker}) = 0$. Since $H_1(B_{n-1}, M_{n-1} \otimes M_{n-1}) = 0$ by part (a) and Proposition 7.5, it suffices to show $H_2(B_{n-1}, M_{n-1} \otimes M_{n-1}) \to H_2(B_{n-1}, M_{n-1})$ is an epimorphism. This follows from Proposition 5.2 and the commutative diagram

$$\begin{array}{ccc} M_{n-1} \otimes M_{n-1} & \to & M_{n-1} \\ \uparrow \rho & \nearrow & \\ V_{n-1} & & \end{array}$$

where we define $\rho(T) = I_{n-1} \otimes T$.

Since $H_2(B_{n-1}, H_0(C_n, V_n \wedge V_n)) = 0$, we may now conclude

$$H_2\big(\operatorname{Gl}(n, p), V_n \wedge V_n\big) \leqslant H_2(B_n, V_n \wedge V_n) \leqslant \mathbf{Z}/p\mathbf{Z}.$$

To conclude that $H_2(\operatorname{Gl}(n, p), V_n \wedge V_n) = \mathbf{Z}/p\mathbf{Z}$, we need only show it is nontrivial.

For this, consider

$$\begin{array}{ccc} H_2(\mathrm{Gl}(2,p),V_2\wedge V_2) & \to & H_2(\mathrm{Gl}(2,p),V_2) \\ \downarrow & & \downarrow \\ H_2(\mathrm{Gl}(n,p),V_n\wedge V_n) & \to & H_2(\mathrm{Gl}(n,p),V_n). \end{array}$$

The upper horizontal arrow is an isomorphism since $V_2 \wedge V_2 \cong V_2$, and the right-hand vertical arrow is an isomorphism by Proposition 5.2. It follows that the lower horizontal arrow is onto something nontrivial, and we are done. (This argument was also suggested by the referee.)

8. Higher multilinear modules.

PROPOSITION 8.1. *For* $n \geqslant 2$, $H_0(B_n, M_n \wedge M_n \otimes M_n) = \mathbf{Z}/p \oplus \mathbf{Z}/p$. *Also,* $H_0(\mathrm{GL}(n,p), V_n \wedge V_n \otimes V_n) = H_0(B_n, V_n \wedge V_n \otimes V_n) = \mathbf{Z}/p\mathbf{Z}$ (*stably*).

PROOF. First note that the exact sequence $0 \to \wedge^2 M_n \otimes V_n \to \wedge^2 M_n \otimes M_n \to \wedge^2 M_n \to 0$ yields

$$H_0\big(B_n, \wedge^2 M_n \otimes V_n\big) \cong H_0\big(B_n, \wedge^2 M_n \otimes M_n\big).$$

In addition, $0 \to \wedge^2 V_n \otimes V_n \to \wedge^2 M_n \otimes V_n \to V_n \otimes V_n \to 0$ yields

$$\begin{aligned} 0 \to H_0\big(B_n, \wedge^2 V_n \otimes V_n\big) &\to H_0\big(B_n, \wedge^2 M_n \otimes V_n\big) \\ &\to H_0(B_n, V_n \otimes V_n) = \mathbf{Z}/p \to 0 \end{aligned}$$

(similarly, for $\mathrm{GL}(n,p)$).

For $n = 2$, $\wedge^2 V_2 \otimes V_2 \cong V_2 \otimes V_2$, and $H_0(B_2, V_2 \otimes V_2) = \mathbf{Z}/p\mathbf{Z}$. The exact sequence above gives the desired result for $\wedge^2 M_2 \otimes M_2$. On the other hand, for $n \geqslant 3$, using those sequences, we need only show $H_0(B_n, \wedge^2 M_n \otimes M_n) = \mathbf{Z}/p \oplus \mathbf{Z}/p$ (stably) (similarly for $\mathrm{GL}(n,p)$).

Assume $n \geqslant 3$. By duality, we have

$$\widehat{H_0(B_n, M_n \wedge M_n \otimes M_n)} \cong H^0(B_n, \widehat{M_n \wedge M_n \otimes M_n}) \cong H^0(B_n, M_n \wedge M_n \otimes M_n).$$

(Recall that M_n is self-dual as noted in the proof of Proposition 5.2.) Since it is technically easier to calculate invariants, we calculate the last group.

Let $z \in M_n \wedge M_n \otimes M_n$ be invariant under B_n. Using the action of the torus, we conclude that z must be of the form

$$\begin{aligned} &\sum_{i\neq j\neq k\neq i} \alpha_{ijk} e_{ij} \wedge e_{jk} \otimes e_{ki} + \sum_{\substack{i<j \\ i\neq k\neq j}} \beta_{ijk} e_{ij} \wedge e_{ji} \otimes e_{kk} \\ &\quad + \sum_{i\neq j\neq k\neq i} \gamma_{ijk} e_{ij} \wedge e_{kk} \otimes e_{ji} + \sum_{i\neq j} \delta_{ij} e_{ij} \wedge e_{ji} \otimes e_{ii} \\ &\quad + \sum_{i\neq j} \varepsilon_{ij} e_{ij} \wedge e_{ii} \otimes e_{ji} + \sum_{i\neq j} \zeta_{ij} e_{ij} \wedge e_{jj} \otimes e_{ji} \\ &\quad + \sum_{\substack{i<j \\ i\neq k\neq j}} \eta_{ijk} e_{ii} \wedge e_{jj} \otimes e_{kk} + \sum_{i\neq j} \theta_{ij} e_{ii} \wedge e_{jj} \otimes e_{ii}. \end{aligned}$$

We now make use of the following argument suggested by the referee. Any z of the *above form* which is invariant under B_n (defined over the the prime field $\mathbf{F}$) is also invariant under $B_n(\bar{\mathbf{F}})$ (defined over the algebraic closure). (Note that $M_n \wedge M_n \otimes M_n$ is contained naturally in the corresponding object defined over the algebraic closure.) Namely, such a z is a sum of terms each of which has weight 0 and hence is invariant under the torus $H_n(\bar{\mathbf{F}})$ over the algebraic closure. If such a z is also B_n-invariant, then it must also be $B_n(\bar{\mathbf{F}})$-invariant because $B_n(\bar{\mathbf{F}})$ is generated by B_n and $H_n(\bar{\mathbf{F}})$. On the other hand, it is well known that the $B_n(\bar{\mathbf{F}})$-invariants are identical with the $\mathrm{GL}_n(\bar{\mathbf{F}})$-invariants. (To see this, note that the orbit of a $B_n(\bar{\mathbf{F}})$-invariant element is the image of the projective variety $\mathrm{Gl}_n(\bar{\mathbf{F}})/B_n(\bar{\mathbf{F}})$ in the affine variety $(M_n \wedge M_n \otimes M_n) \otimes_{\mathbf{F}} \bar{\mathbf{F}}$ and is thus a single point.)

In particular, any B_n-invariant element z of the above form is invariant under the symmetric group Σ_n (contained in $\mathrm{Gl}_n(\mathbf{F})$). Using the transpositions (ij), we conclude that $\beta_{ijk} = \eta_{ijk} = 0$ for all $i \neq j \neq k \neq i$. Using permutations sending i to i', j to j', and k to k', for various triples of indices, we see that $\alpha_{ijk} = \alpha$ is independent of i, j, and k. Similarly, we may write $\gamma_{ijk} = \gamma$, $\delta_{ij} = \delta$, $\varepsilon_{ij} = \varepsilon$, $\zeta_{ij} = \zeta$, and $\theta_{ij} = \theta$ since these coefficients are also independent of the relevant indices.

Consider the coefficient of $e_{ij} \wedge e_{jk} \otimes e_{ii}$ in $x_{ik}(z) - z$ where $i \neq j \neq k \neq i$. Because $\beta_{kji} = 0$, this coefficient is $\alpha_{ijk} - \delta_{ijk}$. Thus, $\alpha = \delta$.

Consider the coefficient of $e_{ij} \wedge e_{ji} \otimes e_{ji}$ in $x_{ji}(z) - z$ where $i \neq j$. It is $\delta_{ij} + \delta_{ji} + \varepsilon_{ij} - \zeta_{ij}$, so that $\zeta = 2\delta + \varepsilon$.

Consider the coefficient of $e_{ij} \wedge e_{jk} \otimes e_{ji}$ in $x_{jk}(z) - z$ where $i \neq j \neq k \neq i$. It is $\alpha_{ijk} - \zeta_{ij} + \gamma_{ijk}$, so that $\gamma = \zeta - \alpha$.

Consider the coefficient of $e_{ii} \wedge e_{ij} \otimes e_{ii}$ in $x_{ij}(z) - z$ where $i \neq j$. It is $\theta_{ij} - \varepsilon_{ij} - \delta_{ij}$, so that $\varepsilon = \theta - \delta$.

Solving these equations yields $\alpha = \delta$, $\zeta = \delta + \theta$, $\gamma = \theta$, and $\varepsilon = -\delta + \theta$, so that we may conclude that a B_n-invariant z (if such exists) must be of the form

$$\begin{aligned}
\delta \sum_{i\neq j\neq k\neq i} e_{ij} \wedge e_{jk} \otimes e_{ki} &+ \theta \sum_{i\neq j\neq k\neq i} e_{ij} \wedge e_{kk} \otimes e_{ji} \\
&+\delta \sum_{i\neq j} e_{ij} \wedge e_{ji} \otimes e_{ii} + (\theta - \delta) \sum_{i\neq j} e_{ij} \wedge e_{ii} \otimes e_{ji} \\
&+ (\theta + \delta) \sum_{i\neq j} e_{ij} \wedge e_{jj} \otimes e_{ji} + \theta \sum_{i\neq j} e_{ii} \wedge e_{jj} \otimes e_{ii} \\
&= \delta \sum_{i,j,k} e_{ij} \wedge e_{jk} \otimes e_{ki} + \theta \sum_{i,j,k} e_{ij} \wedge e_{kk} \otimes e_{ji}.
\end{aligned}$$

Thus, by duality, $H_0(B_n, \wedge^2 M_n \otimes M_n) \leqslant \mathbf{Z}/p \oplus \mathbf{Z}/p$. To see that we have equality (even if B_n is replaced by $\mathrm{GL}(n, p)$, and even stably), consider the (stable) $\mathrm{GL}(n, p)$-invariant forms defined by

$$\phi(A \wedge B \otimes C) = \mathrm{Tr}(A)\mathrm{Tr}(BC) - \mathrm{Tr}(B)\mathrm{Tr}(AC),$$

and

$$\psi(A \wedge B \otimes C) = \mathrm{Tr}(ABC - BAC).$$

These are independent since $\phi(e_{12} \wedge e_{21} \otimes e_{11}) = 0$, $\phi(e_{11} \wedge e_{22} \otimes e_{11}) = -1$, $\psi(e_{12} \wedge e_{21} \otimes e_{11}) = 1$, $\psi(e_{11} \wedge e_{22} \otimes e_{11}) = 0$.

COROLLARY 8.2. *For* $n \geqslant 2$, $H_0(B_n, \wedge^3 M_n) = \mathbf{Z}/p\mathbf{Z}$. *Also,* $H_0(B_n, \wedge^3 V_n) = H_0(\mathrm{GL}(n, p), \wedge^3 V_n) = \mathbf{Z}/p\mathbf{Z}$ *(stably).*

PROOF. Consider the exact sequence $0 \to \wedge^3 V_n \to \wedge^3 M_n \xrightarrow{\lambda} \wedge^2 V_n \to 0$, where $\lambda(A \wedge B \wedge C) = \mathrm{Tr}(A)B \wedge C - \mathrm{Tr}(B)A \wedge C + \mathrm{Tr}(C)A \wedge B$. (Note that the codomain of λ is $\wedge^2 M$, but that in fact $\mathrm{Im}\,\lambda = \wedge^2 V$.)

$$H_0\left(B_n, \wedge^3 V_n\right) \cong H_0\left(B_n, \wedge^3 M_n\right),$$

(and similarly for $\mathrm{GL}(n, p)$), so we need only consider $\wedge^3 V_n$. From Proposition 8.1, it suffices to show the epimorphism

$$\mathbf{Z}/p\mathbf{Z} = H_0\left(B_n, \wedge^2 V_n \otimes V_n\right) \to H_0\left(B_n, \wedge^3 V_n\right)$$

is an isomorphism (similarly for $\mathrm{GL}(n, p)$). However, this is clear since the right-hand side is nontrivial. Indeed, the $\mathrm{GL}(n, p)$-invariant form defined by $\phi(A \wedge B \wedge C) = \mathrm{Tr}(ABC - ACB)$ is nontrivial on $\wedge^3 V_n$ ($\phi(h_1 \wedge e_{12} \wedge e_{21}) = 2$). □

PROPOSITION 8.3. *For* $p \geqslant 5$, $n \geqslant 2$,

$$H_1\left(B_n, \wedge^3 V_n\right) = H_1\left(\mathrm{GL}(n, p), \wedge^3 V_n\right) = 0.$$

PROOF. For $n = 2$, $\wedge^3 V_2 = \mathbf{Z}/p\mathbf{Z} e_{12} \wedge h_1 \wedge e_{21}$.

Since $e_{12} \wedge h_1 \wedge e_{21}$ is fixed by x_{12} and H_2, B_2 acts trivially, and the result follows by Proposition 5.1.

For $n > 2$, we proceed by induction. We form the tower shown as Table 1 (subscripts are omitted). The calculations in the table are done as previously.

One calculation requires further comment.

$$\begin{aligned} H_1\left(B_n, \wedge^2 \hat{W}_{n-1} \otimes W_{n-1}\right) &= H_0\left(B_{n-1}, H_1\left(C_n, \wedge^2 \hat{W}_{n-1} \otimes W_{n-1}\right)\right) \\ &= H_0\left(B_{n-1}, \otimes^2 W \otimes \wedge^2 \hat{W}\right). \end{aligned}$$

However, $\otimes^2 W \otimes \wedge^2 \hat{W}$ is a direct summand of $\otimes^2 W \otimes \otimes^2 \hat{W} \simeq M \otimes M$. Thus $H_0(B_{n-1}, \otimes^2 W \otimes \wedge^2 \hat{W})$ is generated by the images of $e_{11} \otimes e_{11} \simeq e_1 \otimes e_1 \otimes e^1 \otimes e^1 \mapsto e_1 \otimes e_1 \otimes e^1 \wedge e^1 = 0$ and $e_{12} \otimes e_{21} \simeq e_1 \otimes e_2 \otimes e^2 \otimes e^1 \mapsto e_1 \otimes e_2 \otimes e^2 \wedge e^1$.

On the other hand, going the other way, we get

$$\begin{aligned} e_1 \otimes e_2 \otimes e^2 \wedge e^1 &\mapsto \tfrac{1}{2} e_1 \otimes e_2 \otimes \left(e^2 \otimes e^1 - e^1 \otimes e^2\right) \\ &\cong \tfrac{1}{2}\left[e_{12} \otimes e_{21} - e_{11} \otimes e_{22}\right] \\ &\equiv \tfrac{1}{2}\left[e_{12} \otimes e_{21} - e_{11} \otimes e_{11} + e_{12} \otimes e_{21}\right] \\ &= \tfrac{1}{2}\left[2e_{12} \otimes e_{21} - e_{11} \otimes e_{11}\right] \not\equiv 0. \end{aligned}$$

Hence, $H_1(B_n, \wedge^2 \hat{W}_{n-1} \otimes W_{n-1}) = \mathbf{Z}/p\mathbf{Z}$.

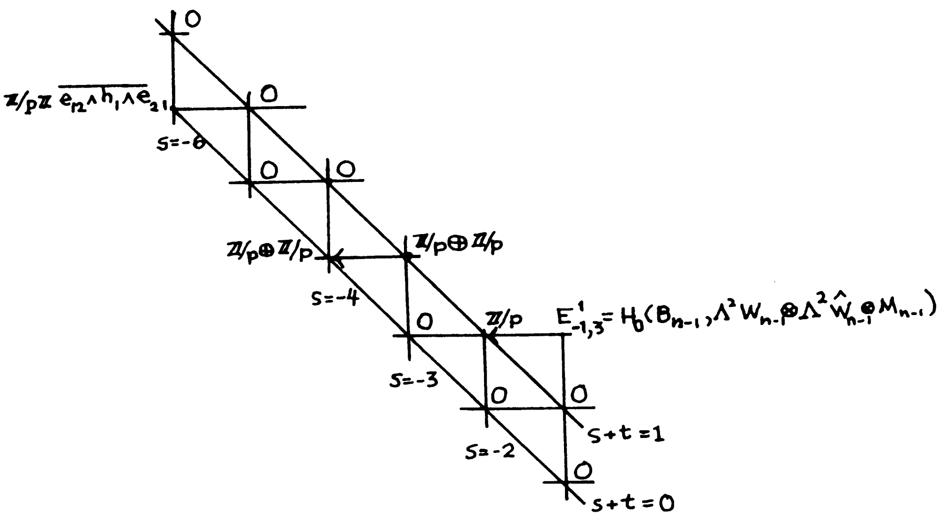
TABLE 1

Level		Factor	$H_r(C_n, -)$	$H_r(B_n, -)$
0	$\Lambda^3 V$			
		$\Lambda^3\hat{W}$	$H_0=H_1=H_2=0$	$H_0=H_1=H_2=0$
1	$\Lambda^2 V \wedge X$			
		$\Lambda^2\hat{W}\otimes M$	$H_0=H_1=0,\ H_2=\Lambda^2 W\otimes\Lambda^2\hat{W}\otimes M$	$H_0=H_1=0$ $H_2=H_0(B_{n-1},\Lambda^2 W\otimes\Lambda^2\hat{W}\otimes M)$
2	$\Lambda^2\hat{W}\otimes X$			
		$\Lambda^2\hat{W}\otimes W$	$H_0=0,\ H_1=\otimes^2 W\otimes\Lambda^2\hat{W}$	$H_0=0$ $H_1=H_0(B_{n-1},\otimes^2 W\otimes\Lambda^2\hat{W})=\mathbb{Z}/p\mathbb{Z}$
3	$V\wedge\Lambda^2 X$			
		$\hat{W}\otimes\Lambda^2 M$	$H_0=0, H_1=M\otimes\Lambda^2 M$	$H_0=0$ $H_1=H_0(B_{n-1},M\otimes\Lambda^2 M)=\mathbb{Z}/p\oplus\mathbb{Z}/p$
4	$\hat{W}\otimes\Lambda^2 X$			
		$\hat{W}\otimes(M\otimes W)$	$H_0=M\otimes M,\ H_1=0$	$H_0=H_0(B_{n-1},M\otimes M)=\mathbb{Z}/p\oplus\mathbb{Z}/p$ $H_1=H_1(B_{n-1},M\otimes M)=0$
5				
		$\hat{W}\otimes\Lambda^2 W$	$H_0=H_1=0$	$H_0=H_1=0$
6	$\Lambda^3 X$			
		$\Lambda^3 M$	$H_0=\Lambda^3 M,\ H_1=0$	$H_0=H_0(B_{n-1},\Lambda^3 M)=\mathbb{Z}/p\mathbb{Z}\ e_{12}\wedge h_1\wedge e_{21}$ $H_1=H_1(B_{n-1},\Lambda^3 M)=0$
7	$\Lambda^2 X\wedge W$			
		$\Lambda^2 M\otimes W$	$H_0=H_1=0$	$H_0=H_1=0$
8	$X\wedge\Lambda^2 W$			
		$M\otimes\Lambda^2 W$	$H_0=H_1=0$	$H_0=H_1=0$
9	$\Lambda^3 W$			
		$\Lambda^3 W$	$H_0=H_1=0$ (For $p=5$, $H_1=W\otimes\Lambda^3 W$)	$H_0=H_1=0$ (including $p=5$)
10	0			

We examine the spectral sequence

$$E^1_{s,t}=H_{s+t}(B_n, A_{-s}/A_{-s+1})\Rightarrow H_{s+t}(B_n, A)$$

as previously. If we exhibit only the range containing nonzero terms, the E^1-term looks like the following



To complete the proof it suffices to show $d^1_{-1,3}$ is an epimorphism. (Then, $d^1_{-3,4}$ is an isomorphism since $H_0(B_n, \wedge^3 V_n) = \mathbf{Z}/p\overline{e_{12} \wedge h_1 \wedge e_{21}}$.)

To show $d^1_{-1,3}$ is an epimorphism, we argue as follows. The corresponding morphism *for* C_n,

$$\begin{array}{ccc} d^1_{-1,3}\colon H_2(C_n, \wedge^2 \hat{W}_{n-1} \otimes M_{n-1}) & \to & H_1(C_n, \wedge^2 \hat{W}_{n-1} \otimes W_{n-1}) \\ \| & & \| \\ \wedge^2 W_{n-1} \otimes \wedge^2 \hat{W}_{n-1} \otimes M_{n-1} & & \otimes^2 W_{n-1} \otimes \wedge^2 \hat{W}_{n-1} = W_{n-1} \otimes (\wedge^2 \hat{W}_{n-1} \otimes W_{n-1}) \end{array}$$

is given by

$$d^1_{-1,3}(u \wedge v \otimes f \wedge g \otimes S) = -[u \otimes (S + \operatorname{Tr} S)v - v \otimes (S + \operatorname{Tr} S)u] \otimes f \wedge g$$

(see below). Since we showed above that

$$H_0(B_{n-1}, \otimes^2 W_{n-1} \otimes \wedge^2 \hat{W}_{n-1}) = \mathbf{Z}/p\mathbf{Z}\, \overline{e_1 \otimes e_2 \otimes e^2 \wedge e^1},$$

it suffices to note that

$$d^1_{-1,3}(e_1 \wedge e_2 \otimes e^2 \wedge e^1 \otimes e_{11}) = -[e_1 \otimes e_2 - 2e_2 \otimes e_1] \otimes e^2 \wedge e^1,$$
$$d^1_{-1,3}(e_1 \wedge e_2 \otimes e^2 \wedge e^1 \otimes e_{22}) = -[2e_1 \otimes e_2 - e_2 \otimes e_1] \otimes e^2 \wedge e^1.$$

(Subtracting twice the second from the first yields $+3e_1 \otimes e_2 \otimes e^2 \wedge e^1$ as required.)

Appendix. Calculation of $d^1_{-1,3}$ for C_n:

$$\begin{aligned} d([u] \cap [v] \otimes \bar{f} \wedge \bar{g} \wedge \bar{S}) &= (d[u] \cap [v] - [u] \cap d[v]) \otimes \bar{f} \wedge \bar{g} \wedge \bar{S} \\ &= [v] \otimes [(-u)(\bar{f} \wedge \bar{g} \wedge \bar{S}) - \bar{f} \wedge \bar{g} \wedge \bar{S}] \\ &\quad - [u] \otimes [(-v)(\bar{f} \wedge \bar{g} \wedge \bar{s}) - \bar{f} \wedge \bar{g} \wedge \bar{S}]. \end{aligned}$$

However, $(-u)\bar{f} = \bar{f} - \overline{(uf)} - (fu)\bar{u}$, similarly for $\bar{g}$, and

$$(-u)\bar{S} = \bar{S} + \overline{S'u}, \qquad S' = S + \operatorname{Tr}(S).$$

Hence, modulo

$$A_3 = V \wedge X \wedge X,\ (-u)(\bar{f} \wedge \bar{g} \wedge \bar{S}) - \bar{f} \wedge \bar{g} \wedge \bar{S} \equiv \bar{f} \wedge \bar{g} \wedge \overline{S'u}.$$

Hence, $d([u] \cap [v] \otimes \bar{f} \wedge \bar{g} \wedge \bar{S}) \equiv [v] \otimes (\bar{f} \wedge \bar{g} \wedge \overline{S'u}) - [u] \otimes (\bar{f} \wedge \bar{g} \wedge \overline{S'v})$. Hence, with some twisting, we get

$$d^1_{-1,3}(u \wedge v \otimes f \wedge g \otimes S) = (v \otimes S'u - u \otimes S'v) \otimes f \wedge g$$

as claimed. □

PROPOSITION 8.4. *For* $n \geqslant 2$, $H_0(B_n, \wedge^4 M_n) = \mathbf{Z}/p\mathbf{Z}$, $H_0(\mathrm{GL}(n, p), \wedge^4 V_n) = H_0(B_n, \wedge^4 V_n) = 0$.

PROOF. We calculate $H^0(B_n, \wedge^4 M_n)$ instead and use duality. Let $z \in M_n \wedge M_n \wedge M_n \wedge M_n$ be invariant under B_n. Using the action of the torus, we conclude that z must be of the form

$$\sum_{\substack{i<j,k\\ k<l\\ k\neq j\neq l}} \alpha_{ijk} e_{ij}\wedge e_{ji}\wedge e_{kl}\wedge e_{lk} + \sum_{\substack{i<j,k,l\\ j\neq k\neq l\neq j}} \beta_{ijkl} e_{ij}\wedge e_{jk}\wedge e_{kl}\wedge e_{li}$$

$$+ \sum_{\substack{j<k\\ j\neq i\neq k}} \gamma_{ijk} e_{ij}\wedge e_{ji}\wedge e_{ik}\wedge e_{ki}$$

$$+ \sum_{\substack{i<j,k\\ i,j,k,l \text{ distinct}}} \delta_{ijkl} e_{ij}\wedge e_{jk}\wedge e_{ki}\wedge e_{ll} + \sum_{i\neq j\neq k\neq i} \varepsilon_{ijk} e_{ij}\wedge e_{jk}\wedge e_{ki}\wedge e_{ii}$$

$$+ \sum_{\substack{i<j\\ k<l\\ i,j,k,l \text{ distinct}}} \zeta_{ijkl} e_{ij}\wedge e_{ji}\wedge e_{kk}\wedge e_{ll} + \sum_{i,j,k \text{ distinct}} \eta_{ijk} e_{ij}\wedge e_{ji}\wedge e_{ii}\wedge e_{kk}$$

$$+ \sum_{i<j} \theta_{ij} e_{ij}\wedge e_{ji}\wedge e_{ii}\wedge e_{jj} + \sum_{i<j<k<l} \lambda_{ijkl} e_{ii}\wedge e_{jj}\wedge e_{kk}\wedge e_{ll}.$$

For $n = 2$ only terms of "type θ" occur. For $n = 3$ only terms of type α, γ, ε, η, and θ occur. However, with minor modifications, the arguments we shall introduce below are also valid in those cases.

As in the proof of Proposition 8.1, we use the referee's argument to conclude that z is in fact $\mathrm{Gl}(n, p)$-invariant and hence invariant under the symmetric group Σ_n. Using the transposition (ij), we conclude that $\alpha_{ijk} = \zeta_{ijk} = \lambda_{ijk} = 0$ for i, j, k distinct. Using the cycle $(ijkl)$, we conclude that $\beta_{ijkl} = 0$ for i, j, k, l distinct. Using permutations sending i to i', j to j', and k to k' for various sets of indices, we conclude that $\gamma_{ijk} = \gamma$, $\delta_{ijk} = \delta$, $\varepsilon_{ijk} = \varepsilon$, $\eta_{ijk} = \eta$, and $\theta_{ijk} = \theta$ are independent of the indices.

Consider the coefficient of $e_{ij}\wedge e_{ji}\wedge e_{ik}\wedge e_{kk}$ in $x_{ik}(z) - z, j < k$. It is $-\gamma_{ijk} - \eta_{ijk} + \varepsilon_{kji}$, so $\gamma = \varepsilon - \eta$.

Consider the coefficient of $e_{ij}\wedge e_{ji}\wedge e_{ki}\wedge e_{ll}$ in $x_{ki}(z) - z, i < j$. It is $\eta_{ijl} - \delta_{ijkl}$, so $\eta = \delta$.

Consider the coefficient of $e_{ij}\wedge e_{ji}\wedge e_{ki}\wedge e_{jj}$ in $x_{ki}(z) - z, i < j$. It is $\theta_{ij} - \eta_{jik} - \varepsilon_{jki}$, so $\theta = \eta + \varepsilon$.

Consider the coefficient of $e_{ij}\wedge e_{jk}\wedge e_{ik}\wedge e_{ki}$ in $x_{ik}(z) - z, j < i$ and $j < k$. It is $-\gamma_{kji} + \varepsilon_{ijk} - \varepsilon_{kij} - \gamma_{ijk}$, so $-2\gamma = 0$.

From these equations we conclude that $\varepsilon = \eta = \delta$ and $\theta = 2\varepsilon$ (with appropriate deletions for $n = 2, 3$). It follows that an invariant z, if such exists, must be of the form given below. (Make the appropriate deletions and modifications for $n = 2, 3$.)

$$\varepsilon\left[\sum_{\substack{i<j,k\\ i,j,k,l \text{ distinct}}} e_{ij}\wedge e_{jk}\wedge e_{ki}\wedge e_{ll} + \sum_{i,j,k \text{ distinct}} e_{ij}\wedge e_{ji}\wedge e_{ki}\wedge e_{ii}\right.$$

$$\left. + \sum_{i,j,k \text{ distinct}} e_{ij}\wedge e_{ji}\wedge e_{ii}\wedge e_{kk} + 2\sum_{i<j} e_{ij}\wedge e_{ji}\wedge e_{ii}\wedge e_{jj}\right]$$

$$= \delta\left[\sum_{\substack{i<j,k\\ j\neq k}} e_{ij}\wedge e_{jk}\wedge e_{ki}\wedge e_{ll} + \sum_{i\neq j} e_{ij}\wedge e_{ji}\wedge e_{ii}\wedge e_{kk}\right].$$

Thus, by duality, $H_0(B_n, \wedge^4 M_n) \leqslant \mathbf{Z}/p\mathbf{Z}$. To complete the proof, consider the exact sequence $0 \to \wedge^4 V_n \to \wedge^4 M_n \xrightarrow{\lambda} \wedge^3 V_n \to 0$ where

$$\lambda(A_1 \wedge A_2 \wedge A_3 \wedge A_4) = \sum_{i=1}^{4} (-1)^{i+1}\mathrm{Tr}(A_i)A_1 \wedge \cdots \wedge \widehat{A_i} \wedge \cdots \wedge A_4.$$

(Note as in the proof of Corollary 8.2 that λ initially takes its values in $\wedge^3 M_n$, but it is possible to show that $\mathrm{Im}\,\lambda = \wedge^3 V_n$.) Since $H_1(B_n, \wedge^3 V_n) = 0$, $H_0(B_n, \wedge^3 V_n) = \mathbf{Z}/p\mathbf{Z}$, it follows that $H_0(B_n, \wedge^4 M_n) = \mathbf{Z}/p\mathbf{Z}$ and $H_0(B_n, \wedge^4 V_n) = 0$. □

CHAPTER III. CALCULATIONS OF d^2 FOR $n = 2$

9. The relevant p-groups. We now turn to the needed calculations in the spectral sequence for $n = 2$. We start by considering the spectral sequence of the group extension $1 = V \to \overline{U} \to P \to 1$ where (a) $\overline{U}$ is the p-Sylow subgroup of $\overline{\mathrm{SL}}(2, p^2)$ consisting of all matrices over $\mathbf{Z}/p^2\mathbf{Z}$ of the form

$$\begin{bmatrix} 1+pa & b \\ pc & 1+pd \end{bmatrix}, \qquad a + d - bc \equiv 0 \mod p,$$

(b) $P = U_2$ consists of all matrices over $\mathbf{Z}/p\mathbf{Z}$ of the form $\left[\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right]$, and (c) $V = V_2$ as before.

Using multiplicative notation, we choose the following generators:

For P,

$$y = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

(Let $\bar{y} = \left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]$ in $\overline{U}$.)

For V,

$$z_1 = \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix} = 1 + pE_{21} \quad \text{where } \overline{E}_{21} = e_{21},$$

$$z_2 = \begin{bmatrix} 1+p & 0 \\ 0 & 1-p \end{bmatrix} = 1 + pH_1 \quad \text{where } \overline{H}_1 = h_1,$$

$$z_3 = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} = 1 + pE_{12} \quad \text{where } \overline{E}_{12} = e_{12}.$$

Then,

(9.1) $$\bar{y}^{-1}z_1\bar{y} = z_1z_2z_3^{-1}, \quad \bar{y}^{-1}z_2\bar{y} = z_2z_3^{-2}, \quad \bar{y}^{-1}z_3\bar{y} = z_3.$$

The calculations will be simpler if we change the basis for V to $x_1 = z_1$, $x_2 = z_2z_3^{-1}$, $x_3 = z_3^{-2}$. Then

(9.2) $$\bar{y}^{-1}x_1\bar{y} = x_1x_2, \quad \bar{y}^{-1}x_2\bar{y} = x_2x_3, \quad \bar{y}^{-1}x_3\bar{y} = x_3.$$

(Additively, y^{-1} has matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

on V.) Since the Charlap-Vasquez theory is for cohomology, we shall make our spectral sequence calculations for cohomology with coefficients in $\mathbf{F} = \mathbf{Z}/p\mathbf{Z}$ and dualize later. Thus we shall need the dual bases. Define $\alpha_i(x_j) = \delta_{ij}$. Then,

$$y\alpha_1 = \alpha_1, \quad y\alpha_2 = \alpha_1 + \alpha_2, \quad y\alpha_3 = \alpha_2 + \alpha_3. \tag{9.3}$$

10. The Charlap-Vasquez description of d_2. Let $1 \to V \to U \to P \to 1$ be a group extension with V a finitely generated abelian group. Charlap and Vasquez **[C-V]** have described

$$d_2^{s,t}\colon E_2^{s,t}(A) \to E_2^{s+2,t-1}(A)$$

for A any $\mathbf{F}(U)$-module on which V acts trivially. ($\mathbf{F}$ should be a PID and each $H_t(V,\mathbf{F})$ should be $\mathbf{F}$-free.) We shall use their description in the case *V is an elementary abelian p-group, $\mathbf{F} = \mathbf{Z}/p\mathbf{Z}$, and $A = \mathbf{F}$.*

In order to state the Charlap-Vasquez result, we need to introduce some terminology involving certain standard identifications. We have as usual

$$H^t(V,\mathbf{F}) = \widehat{H_t(V,\mathbf{F})}.$$

(In particular, for $\alpha \in H^t(V,\mathbf{F})$, $\mu \in H_t(V,\mathbf{F})$, $\alpha(\mu)$ is calculated by evaluating a cochain representing α on a chain representing μ.) Moreover, $H^{t-1}(V, H_t(V,\mathbf{F})) = H^{t-1}(V,\mathbf{F}) \otimes H_t(V,\mathbf{F})$, so we have a pairing q: $H^t(V,\mathbf{F}) \otimes H^{t-1}(V, H_t(V,\mathbf{F})) \to H^{t-1}(V,\mathbf{F})$ defined by

$$q(\alpha \otimes (\beta \otimes \mu)) = \alpha(\mu)\beta. \tag{10.0}$$

q in turn induces a pairing

$$\cup_q\colon H^s(P, H^t(V,\mathbf{F})) \otimes H^2(P, H^{t-1}(V, H_t(V,\mathbf{F}))) \to H^{s+2}(P, H^{t-1}(V,\mathbf{F})),$$

that is, $\cup_q$: $E_2^{s,t}(\mathbf{F}) \otimes E_2^{2,t-1}(H_t(V,\mathbf{F})) \to E_2^{s+2,t-1}(\mathbf{F})$. Then Charlap and Vasquez assert that for $\xi \in E_2^{s,t}(\mathbf{F})$,

$$d_2^{s,t}(\xi) = (-1)^s \xi \cup_q \left(V^t - Q_*(\chi)\right) \tag{10.1}$$

where $\chi \in H^2(P,V)$ is the class of the extension, Q: $V \to H^{t-1}(V, H_t(V,\mathbf{F}))$ is derived from the Pontryagin product $\mu \cap \nu$ in $H_*(V,\mathbf{F})$, and $V^t \in H^2(P, H^{t-1}(V, H_t(V,\mathbf{F})))$ is a certain universal characteristic class.

Charlap and Vasquez (and later Sah [S]) have investigated V^t for $\mathbf{F} = \mathbf{Z}$ and V free abelian, but little is known about it in our case. We shall rely simply on the fact that for the corresponding split extension ($\chi = 0$)

$$d_2^{\mathrm{spl}}(\xi) = (-1)^s \xi \cup_q V^t.$$

Thus, we obtain the *first* term in (10.1) if we can calculate $d_2^{\mathrm{spl}}(\xi)$ by some other method. This is done in §11.

We concentrate now on calculating the second term $(-1)^{s+1}\xi \cup_q Q_*(\chi)$. Let $\mu_1, \mu_2, \ldots, \mu_n$ be a basis for $H_{t-1}(V,\mathbf{F})$, $\gamma_1, \gamma_2, \ldots, \gamma_n$ the dual basis for $H^{t-1}(V,\mathbf{F})$, and $\zeta_1, \zeta_2, \ldots, \zeta_m$ a basis for $H_t(V,\mathbf{F})$. For $z \in H_1(V,\mathbf{F}) = V$, let $\mu_i \cap z = \sum_{l=1}^m a_{il}\zeta_l$.

Then one may translate the definition of Q in Charlap-Vasquez (called P there) into the formula:

(10.2) $$Q(z) = \sum_{i,l} a_{il}\gamma_i \otimes \zeta_l \in H^{t-1}(V,\mathbf{F}) \otimes H_t(V,\mathbf{F}).$$

We shall be especially interested in the case $V = V_2$ (as in §9), $t = 1,2,3$. In order to calculate Q_* explicitly, we need bases and dual bases in the relevant dimensions. We start with a description of the Pontryagin product in terms of standard chains. (See the table of notations for definitions of these terms.)

PROPOSITION 10.3. *Let X be the standard chain complex over $\mathbf{Z}$ for the abelian group V. The Pontryagin product is induced by a map (of complexes over $\mathbf{Z}$) $\cap: X \otimes X \to X$ which is consistent with the group homomorphism $V \times V \to V$ defined by multiplication. ($X \otimes X$ is viewed as a $V \times V$-complex by componentwise action.) That map necessarily satisfies the rule $d(x \cap y) = dx \cap y + (-1)^{\deg x} x \cap dy$. One may choose this map of complexes so that in degrees* 0, 1, 2, *and* 3 *it is given by*

$$\begin{gathered}
[\cdot] \cap [\cdot] = [\cdot], \\
[u] \cap [\cdot] = [\cdot] \cap [u] = [u], \\
[u,v] \cap [\cdot] = [\cdot] \cap [u,v] = [u,v], \\
[u] \cap [v] = [u,v] - [v,u], \\
[u,v,w] \cap [\cdot] = [\cdot] \cap [u,v,w] = [u,v,w], \\
[u] \cap [v,w] = [u,v,w] - [v,u,w] + [v,w,u], \\
[u,v] \cap [w] = [u,v,w] - [u,w,v] + [w,u,v].
\end{gathered}$$

PROOF. The Pontryagin product may be defined as the composite map $H_*(V,\mathbf{F}) \otimes H_*(V,\mathbf{F}) \to H_*(V \times V,\mathbf{F}) \to H_*(V,\mathbf{F})$ where the first component is the Künneth isomorphism and the second is induced by the group homomorphism $V \times V \to V$. To compute it on the chain level, we note that $X \otimes X$ is a $V \times V$-resolution of $\mathbf{Z}$. Hence, we need only define a map of complexes $X \otimes X \to X$ (consistent with augmentations and differentials) which is consistent with the group homomorphism $V \times V \to V$. (The map is unique up to chain homotopy.) Since the differential in $X \otimes X$ is given by $d(x \otimes y) = dx \otimes y + (-1)^{\deg x} x \otimes dy$, the map is necessarily a derivation as stated. The main content of the proposition is that the chain map defined above in low degrees has the right properties. One checks this by direct calculations. For example, $d_1([\cdot] \cap [u]) = d_1([u]) = u[\cdot] - [\cdot]$, and

$$\begin{aligned}
\cap\, ((d_0 \otimes \mathrm{id} + \mathrm{id} \otimes d_1)([\cdot] \otimes [u])) &= \cap\, ([\cdot] \otimes d_1[u]) \\
&= \cap\, ([\cdot] \otimes (u[\cdot] - [\cdot])) = [\cdot] \cap u[\cdot] - [\cdot] \cap [\cdot] \\
&= u([\cdot] \cap [\cdot]) - [\cdot] \cap [\cdot] = u[\cdot] - [\cdot].
\end{aligned}$$

The other calculations are similarly tedious and we omit them. Note that the map of complexes has been defined on a $Z[V \times V]$-basis for $X \otimes X$, and semilinearity with respect to $V \times V \to V$ has been used extensively. □

Recall the notation $\rho(x) \in H_2(V,\mathbf{F})$ for $x \in V$, $\delta\alpha \in H^2(V,\mathbf{F})$ for $\alpha \in \hat{V}$.

PROPOSITION 10.4. *Let $V = V_2$, and choose a basis x_1, x_2, x_3 for $V = H_1(V, \mathbf{F})$ and dual basis $\alpha_1, \alpha_2, \alpha_3$ for $\hat{V} = H^1(V, \mathbf{F})$. Then the following are bases and dual bases for $H_t(V, \mathbf{F})$ and $H^t(V, \mathbf{F})$ respectively.*

For $t = 2$, $x_m \cap x_n$, $m < n$, $\rho(x_m)$, $m = 1, 2, 3$, with dual bases $\alpha_m\alpha_n$, $m < n$, $\delta\alpha_m$, $m = 1, 2, 3$.

For $t = 3$, $x_l \cap x_m \cap x_n$, $l < m < n$, $x_l \cap \rho(x_m)$, $l, m = 1, 2, 3$ with dual bases $\alpha_l\alpha_m\alpha_n$, $l < m < n$, $\alpha_l\delta\alpha_m$, $l, m = 1, 2, 3$.

PROOF. Use the formulas in Proposition 10.3 to evaluate the cohomology classes on the homology classes. For example, $(\alpha_l\delta\alpha_m)(x_l \cap \rho(x_m))$ is calculated as follows. Let a_l denote α_l viewed as a 1-cocycle on V, and let δa_l be the standard cocycle representing $\delta\alpha_l$. Then $x_l \cap \rho(x_m)$ is represented by

$$\sum_{i=0}^{p-1} [x_l] \cap [x_m, ix_m] = \sum_{i=0}^{p-1} ([x_l, x_m, ix_m][x_m, x_l, ix_m] + [x_m, ix_m, x_l])$$

and

$$(\alpha_l\delta\alpha_m)(x_l \cap \rho(x_m))$$
$$= \sum_{i=0}^{p-1} (a_l(x_l)\delta a_m(x_m, ix_m) - a_l(x_m)\delta a_m(x_l, ix_m) + a_l(x_m)\delta a_m(ix_m, x_l)).$$

If $l \neq m$, there is only one nonzero term, the first term with $i = p - 1$. If $l = m$, all terms for $i = p - 1$ are nonzero, but two cancel. In any event, the net result is 1.

The other cases are similar. □

PROPOSITION 10.5. *Let $1 \to V \to \overline{U} \to P \to 1$ be the extension in §9, and let $p > 3$. Then choosing y as generator for P, we have $H^2(P, V) = V^P = \mathbf{Z}/p\mathbf{Z}x_3$ and the class of the extension χ is given by $\frac{1}{2}x_3$.*

PROOF. Generally, $H^2(P, V) = V^P/TV$ where $T = 1 + y + y^2 + \cdots + y^{p-1}$. In our case, it is easy to see that $TV = 0$. To determine the class of the extension, let

$$\bar{y} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \text{ in } \overline{\mathrm{SL}}(2, p^2).$$

Then

$$\bar{y}^p = \begin{bmatrix} 1 & -p \\ 0 & 1 \end{bmatrix} = z_3^{-1}.$$

Consider now the commutative diagram:

$$\begin{array}{ccccccccc} 1 & \to & Gp\langle z_3^{-1}\rangle = \mathbf{Z}/p\mathbf{Z} & \to & Gp\langle \bar{y}\rangle = \mathbf{Z}/p^2\mathbf{Z} & \to & Gp\langle y\rangle = \mathbf{Z}/p\mathbf{Z} & \to & 1 \\ & & \downarrow \phi & & \downarrow & & \| & & \\ 1 & \to & V & \to & \overline{U} & \to & P & \to & 1 \end{array}$$

The class of the upper extension is represented by z_3^{-1}; hence so is that of the lower extension.

Changing to additive notation, we may identify χ with $-z_3 = \frac{1}{2}x_3$. □

PROPOSITION 10.6. *Let P and V be as in* §9, *and let $p > 3$. Then*
(i) $H^1(V,\mathbf{F})^P = \mathbf{F}\alpha_1$,
(ii) $H^2(V,\mathbf{F})^P = \mathbf{F}\alpha_1\alpha_2 \oplus \mathbf{F}\delta\alpha_1$,
(iii)

$$H^3(V,\mathbf{F})^P = \mathbf{F}\alpha_1\alpha_2\alpha_3 \oplus \mathbf{F}\alpha_1\delta\alpha_1 \oplus \mathbf{F}(\alpha_1\delta\alpha_2 - \alpha_2\delta\alpha_1)$$
$$\oplus \mathbf{F}(\alpha_1\delta\alpha_3 + \alpha_3\delta\alpha_1 - \alpha_2\delta\alpha_2 + \alpha_2\delta\alpha_1).$$

PROOF. (i) is clear. (ii) follows easily since $H^2(V,\mathbf{F}) = \wedge^2\hat{V} + \delta\hat{V}$ (as an $\mathbf{F}(P)$-module), and $\wedge^2\hat{V} \cong \hat{V}$.

To prove (iii), note first that $H^3(V,\mathbf{F}) = \wedge^3\hat{V} \oplus \hat{V}\delta\hat{V}$ (as an $\mathbf{F}(P)$-module). Moreover, $\wedge^3\hat{V} \cong \mathbf{F}$, and

$$\hat{V}\delta\hat{V} \cong \hat{V} \otimes \hat{V} = \wedge^2\hat{V} \oplus \hat{V} \circ \hat{V}.$$

Again, $\wedge^2\hat{V} = \hat{V}$ and a basic invariant is $\alpha_1 \wedge \alpha_2$. Tracing back through the identifications yields first $\alpha_1 \otimes \alpha_2 - \alpha_2 \otimes \alpha_1$ and then $\alpha_1\delta\alpha_2 - \alpha_2\delta\alpha_1$.

To calculate $(\hat{V} \circ \hat{V})^P$, let $\gamma = \sum_{i=1}^3 b_i\alpha_i \circ \alpha_i + \sum_{i<j} b_{ij}\alpha_i \circ \alpha_j$ be invariant. Then

$$\begin{aligned} y(\gamma) = {} & b_1\alpha_1 \circ \alpha_1 + b_2\alpha_2 \circ \alpha_2 + 2b_2\alpha_1 \circ \alpha_2 + b_2\alpha_1 \circ \alpha_1 \\ & + b_3\alpha_3 \circ \alpha_3 + 2b_3\alpha_2 \circ \alpha_3 + b_3\alpha_2 \circ \alpha_2 \\ & + b_{12}\alpha_1 \circ \alpha_1 + b_{12}\alpha_1 \circ \alpha_2 + b_{23}\alpha_1 \circ \alpha_2 + b_{23}\alpha_2 \circ \alpha_2 \\ & + b_{23}\alpha_1 \circ \alpha_3 + b_{23}\alpha_2 \circ \alpha_3 + b_{13}\alpha_1 \circ \alpha_2 + b_{13}\alpha_1 \circ \alpha_3. \end{aligned}$$

Hence, $b_2 + b_{12} = b_3 + b_{23} = 2b_2 + b_{23} + b_{13} = 2b_3 = b_{23} = 0$. Thus, $b_3 = b_{23} = 0$, $b_{12} = -b_2$, $b_{13} = -2b_2$, and

$$\gamma = b_1\alpha_1 \circ \alpha_1 + b_2(\alpha_2 \circ \alpha_2 - \alpha_1 \circ \alpha_2 - 2\alpha_1 \circ \alpha_3).$$

Tracing through the identifications yields $\alpha_1 \circ \alpha_1 \mapsto 2\alpha_1\delta\alpha_1$ and $\alpha_2 \circ \alpha_2 - \alpha_1 \circ \alpha_2 - 2\alpha_1 \circ \alpha_3 \mapsto 2\alpha_2\delta\alpha_2 - \alpha_1\delta\alpha_2 - \alpha_2\delta\alpha_1 - 2\alpha_1\delta\alpha_3 - 2\alpha_3\delta\alpha_1$. Adding $\alpha_1\delta\alpha_2 - \alpha_2\delta\alpha_1$ yields

$$2(\alpha_2\delta\alpha_2 - \alpha_2\delta\alpha_1 - \alpha_1\delta\alpha_3 - \alpha_3\delta\alpha_1)$$

as required. □

PROPOSITION 10.7. *Let $1 \to V \to \overline{U} \to P \to 1$ be the extension in* 10.5, *and assume $p > 3$. The second term $\Delta = (-1)^{s+1}\xi \cup_q (Q_*(\chi))$ in the Charlap-Vasquez decomposition* (10.1) *is given as follows* ($s = 0$ *in all cases*):
(i)

$$\xi = \alpha_1: \qquad \Delta = 0 \quad \textit{in } H^2(P,\mathbf{F}) = \mathbf{F},$$

(ii)

$$\left.\begin{aligned} \xi = \alpha_1\alpha_2: &\quad \Delta = 0 \\ \xi = \delta\alpha_1: &\quad \Delta = 0 \end{aligned}\right\} \quad \textit{in } H^2(P,\hat{V}) = \hat{V}^P,$$

(iii)

$$\left.\begin{array}{ll} \xi = \alpha_1\alpha_2\alpha_3: & \Delta = -\frac{1}{2}\alpha_1\alpha_2 \\ \xi = \alpha_1\delta\alpha_1: & \Delta = 0 \\ \xi = \alpha_1\delta\alpha_2 - \alpha_2\delta\alpha_1: & \Delta = 0 \\ \xi = \alpha_1\delta\alpha_3 + \alpha_3\delta\alpha_1 - \alpha_2\delta\alpha_2 + \alpha_2\delta\alpha_1: & \Delta = -\frac{1}{2}\delta\alpha_1 \end{array}\right\}$$

$$\textit{in } H^2(P, H^2(V,\mathbf{F})) = \left(\wedge^2 \hat{V}\right)^P \oplus (\delta\hat{V})^P. \quad \square$$

PROOF. Recall first that given a P-pairing q: $A \otimes B \to C$, if $\alpha \in H^s(P, A)$ is represented by $a \in A$ and $\beta \in H^t(P, B)$ is represented by $b \in B$, then $\alpha \cup_q \beta \in H^{s+t}(P, C)$ is represented by $q(a \otimes b)$ *provided* s or t is even. (See **[C-E**, Chapter XII, §7].)

Abbreviate $z = \frac{1}{2}x_3$, and use the bases and dual bases given in Proposition 10.4

(i)

$$1 \cap z = \tfrac{1}{2}x_3, \text{ so } Q(z) = \tfrac{1}{2}1 \otimes x_3 \ \left(1 \in H^0(V,\mathbf{F})\right) \ (\text{use } (10.2)).$$

Hence (from (10.0)),

$$\alpha_1 \cup_q Q_*(\chi) = \tfrac{1}{2}q(\alpha_1 \otimes (1 \otimes x_3)) = \tfrac{1}{2}\alpha_1(x_3) = 0.$$

(ii) $x_1 \cap z = \frac{1}{2}x_1 \cap x_3$, $x_2 \cap z = \frac{1}{2}x_2 \cap x_3$, $x_3 \cap z = 0$, so

$$Q(z) = \tfrac{1}{2}(\alpha_1 \otimes x_1 \cap x_3 + \alpha_2 \otimes x_2 \cap x_3).$$

Hence, $\alpha_1\alpha_2 \cup_q Q_*(\chi) = \frac{1}{2}[q(\alpha_1\alpha_2 \otimes (\alpha_1 \otimes x_1 \cap x_3)) + q(\alpha_1\alpha_2 \otimes (\alpha_2 \otimes x_2 \cap x_3))]$ $= 0$, and $\delta\alpha_1 \cup_q Q_*(\chi) = 0$.

(iii) $x_1 \cap x_2 \cap z = \frac{1}{2}x_1 \cap x_2 \cap x_3$, $x_1 \cap x_3 \cap z = x_2 \cap x_3 \cap z = 0$, $\rho(x_i) \cap z = \frac{1}{2}x_3 \cap \rho(x_i)$, $i = 1, 2, 3$, so

$$Q(z) = \frac{1}{2}\left[\alpha_1\alpha_2 \otimes x_1 \cap x_2 \cap x_3 + \sum_{i=1}^{3} \delta\alpha_i \otimes x_3 \cap \rho(x_i)\right].$$

Hence,

$$\begin{aligned} \alpha_1\alpha_2\alpha_3 \cup_q Q_*(\chi) &= \tfrac{1}{2}q(\alpha_1\alpha_2\alpha_3) \otimes (\alpha_1\alpha_2 \otimes x_1 \cap x_2 \cap x_3) \\ &= \tfrac{1}{2}\alpha_1\alpha_2\alpha_3(x_1 \cap x_2 \cap x_3)\alpha_1\alpha_2 = \tfrac{1}{2}\alpha_1\alpha_2, \end{aligned}$$

$$\alpha_1\delta\alpha_1 \cup_q Q_*(\chi) = 0, \qquad (\alpha_1\delta\alpha_2 - \alpha_2\delta\alpha_1) \cup_q Q_*(\chi) = 0,$$

$$\begin{aligned} &(\alpha_1\delta\alpha_3 + \alpha_3\delta\alpha_1 - \alpha_2\delta\alpha_2 + \alpha_2\delta\alpha_1) \cup_q Q_*(\chi) \\ &\qquad = \tfrac{1}{2}\left[(\alpha_3\delta\alpha_1)(x_3 \cap \rho(x_1))\delta\alpha_1\right] = \tfrac{1}{2}\delta\alpha_1. \quad \square \end{aligned}$$

11. d_2 for the split extension. Let $\tilde{U} = P \ltimes V$. Write V additively, and denote the action of P on V by $y(v)$. We shall calculate $d_2^{\mathrm{spl}}(\xi)$ by making use of explicit resolutions. Let $X = \bigoplus \mathbf{Z}[V][v_1, v_2, \ldots, v_n]$ be the standard $\mathbf{Z}[V]$-free resolution of $\mathbf{Z}$. Recall that

$$\begin{aligned} d_V[v_1, v_2, \ldots, v_n] = v_1[v_2, \ldots, v_n] &+ \sum_{i=1}^{n-1} (-1)^i [v_1, \ldots, v_i + v_{i+1}, \ldots, v_n] \\ &+ (-1)^n [v_1, v_2, \ldots, v_{n-1}] \end{aligned}$$

and $\varepsilon_v([\cdot]) = 1$. Note also that P acts on X by $y(v[v_1, v_2, \ldots, v_n]) = y(v)[y(v_1), y(v_2), \ldots, y(v_n)]$. Also, $d_V y(x) = y(d_V x)$, $\varepsilon(y(x)) = \varepsilon(x)$, and $y(vx) = y(v)y(x)$. Let Y be the usual minimal $\mathbf{Z}[P]$-free resolution of $\mathbf{Z}$. In particular, $Y_n = \mathbf{Z}[P]y_n$, $d_P y_{2n} = Ty_{2n-1}$, $d_P y_{2n-1} = (y-1)y_{2n-2}$ $(n > 0)$, and $\varepsilon_P y_0 = 1$. $(T = 1 + y + y^2 + \cdots + y^{p-1}.)$

Make $Y \otimes X$ into a $\mathbf{Z}[P \ltimes V]$-resolution of $\mathbf{Z}$ as usual. In particular, define $y(y_s \otimes x_t) = y(y_s) \otimes y(x_s)$, $v(y_s \otimes x_t) = y_s \otimes vx_t$.

PROPOSITION 11.1. *$Y \otimes X$ is a $\mathbf{Z}[\tilde{U}]$-free resolution of $\mathbf{Z}$.*

PROOF. We have an isomorphism of $\mathbf{Z}[\tilde{U}]$-modules $\mathbf{Z}[\tilde{U}] \otimes_V X_t \cong \mathbf{Z}[P] \otimes X_t \cong Y_s \otimes X_t$ defined by $vy \otimes x \mapsto y \otimes v(y(x))$. The left-hand side is $\mathbf{Z}[\tilde{U}]$-free since X_t is $\mathbf{Z}[V]$-free. □

It follows that $H^*(\tilde{U}, \mathbf{F})$ is the homology of the complex

$$C = \operatorname{Hom}_{\mathbf{Z}[\tilde{U}]}(Y \otimes X, \mathbf{F}) \cong \operatorname{Hom}_{\mathbf{Z}[P]}(Y, \operatorname{Hom}_{\mathbf{Z}[V]}(X, \mathbf{F})).$$

C is in fact a double complex with differentials defined as follows: Since $Y_s = \mathbf{Z}[P]y_s$, we may identify $C^{s,t} = \operatorname{Hom}_P(\mathbf{Z}[P], \operatorname{Hom}_V(X_t, \mathbf{F})) = \operatorname{Hom}_V(X_t, \mathbf{F})$. Then $d_{\mathrm{I}}^{s,t} = (-1)^s d_V^t$ and

$$d_{\mathrm{II}} f = \begin{cases} (y-1)f, & s \text{ even}, s > 0, \\ Tf, & s \text{ odd}. \end{cases}$$

As usual, $d_{\mathrm{I}} d_{\mathrm{II}} + d_{\mathrm{II}} d_{\mathrm{I}} = 0$.

The double complex yields a spectral sequence with $E_2^{s,t} = H^s(P, H^t(V, \mathbf{F})) = H_{\mathrm{II}}^s(H_{\mathrm{I}}^t(C))$ and limit $H^*_{\mathrm{Total}}(C) = H^*(\tilde{U}, \mathbf{F})$.

PROPOSITION 11.2. *The spectral sequence is the Lyndon-Hochschild-Serre spectral sequence of the split group extension* $1 \to V \to \tilde{U} \to P \to 1$.

PROOF. The Lyndon-Hochschild-Serre spectral sequence may be realized as the spectral sequence of the double complex (as above) $\operatorname{Hom}_{\mathbf{Z}(P)}(Y, \operatorname{Hom}_{\mathbf{Z}(V)}(W, \mathbf{F}))$ where W is any $\mathbf{Z}(\tilde{U})$-projective resolution of $\mathbf{Z}$ (see Evens [**E**]). Since X is an acyclic $\mathbf{Z}(\tilde{U})$-complex over $\mathbf{Z}$, there is a map of $\mathbf{Z}(\tilde{U})$-complexes over $\mathbf{Z}$, G: $W \to X$ (and G is unique up to $\mathbf{Z}(\tilde{U})$-chain homotopy). Since such a G is also a $\mathbf{Z}(V)$-map, and since X and W are projective $\mathbf{Z}(V)$-resolutions of $\mathbf{Z}$, G is an isomorphism up to $\mathbf{Z}(V)$-chain homotopy. G induces a morphism of double complexes

$$\operatorname{Hom}_{\mathbf{Z}(P)}(Y, \operatorname{Hom}_{\mathbf{Z}(V)}(X, \mathbf{F})) \to \operatorname{Hom}_{\mathbf{Z}(P)}(Y, \operatorname{Hom}_{\mathbf{Z}(V)}(W, \mathbf{F}))$$

which by the above remark yields an isomorphism (the identity)

$$\operatorname{Hom}_{\mathbf{Z}(P)}(Y, H^*(V, \mathbf{F})) \to \operatorname{Hom}_{\mathbf{Z}(P)}(Y, H^*(V, \mathbf{F}))$$

on the E_1-level (and hence for E_r, $r \geqslant 1$).

Arguing for W exactly as for X, we see $Y \otimes W$ is also a $\mathbf{Z}(\tilde{U})$-projective resolution of $\mathbf{Z}$ and $\mathrm{id} \otimes G$: $Y \otimes W \to Y \otimes X$ induces the identity $H^*(\tilde{U}, \mathbf{F}) \to H^*(\tilde{U}, \mathbf{F})$ for cohomology of the two corresponding *total* complexes $\operatorname{Hom}_{\mathbf{Z}(P)}(Y, \operatorname{Hom}_{\mathbf{Z}(V)}(X, \mathbf{F}))$ and $\operatorname{Hom}_{\mathbf{Z}(P)}(Y, \operatorname{Hom}_{\mathbf{Z}(V)}(W, \mathbf{F}))$. □

To calculate $d_2^{0,t}\alpha$ for $\alpha \in H^t(V,\mathbf{F})^P$, proceed as follows. Choose a cocycle $a \in \operatorname{Hom}_V(X_t,\mathbf{F})$ representing α. Since $y\alpha = \alpha$, we have $(y-1)a = d_V b$ where $b \in \operatorname{Hom}_V(X_{t-1},\mathbf{F})$. Then

$$\begin{aligned} d(a+b) &= (d_{\mathrm{I}} + d_{\mathrm{II}})(a+b) = d_V a + (y-1)a + (-1)d_V b + Tb \\ &= 0 + (y-1)a - d_V b + Tb = Tb. \end{aligned}$$

Hence,

(11.3) $a + b$ represents $\alpha \in E_2^{0,t}$ and $d(a+b) = Tb$ represents $d_2\alpha \in E_2^{2,t-1}$.

We are now ready to calculate $d_2^{0,t}$ for $t = 1, 2, 3$. First, $d_2^{0,1} = 0$ since the splitting implies $E_2^{*,0} = H^*(P,\mathbf{F})$ is a direct summand of $H^*(\tilde{U},\mathbf{F})$.

PROPOSITION 11.4. *For $p > 3$,*

$$d_2^{0,2}(\alpha_1\alpha_2) = 0, \qquad d_2^{0,2}\delta\alpha_1 = 0.$$

PROOF. Let a_i be identical with α_i but viewed as a 1-cocycle on V. Then a_1a_2 represents $\alpha_1\alpha_2$, and $ya_1 = a_1$, $ya_2 = a_1 + a_2$, $ya_3 = a_2 + a_3$, so $y(a_1a_2) = a_1^2 + a_1a_2$. Hence, $(y-1)(a_1a_2) = a_1^2$. Moreover, $a_1^2 = d_V u_1$ where $u_1(v) = -\frac{1}{2}i_1^2$ for $v = i_1x_1 + i_2x_2 + i_3x_3 \in V$. Hence, $d_2(\alpha_1\alpha_2)$ is represented by Tu_1. However, by 9.2, $y^{-1}(v) = i_1x_1 + (i_1 + i_2)x_2 + (i_2 + i_3)x_3$, so $yu_1(v) = -\frac{1}{2}i_1^2$ and $yu_1 = u_1$. Hence, $Tu_1 = pu_1 = 0$.

To show $d_2\delta\alpha_1 = 0$, we use $d_2\alpha_1 = 0$ and the following result.

LEMMA 11.5. *The diagram below anticommutes.*

$$\begin{array}{ccc} H^s(P, H^t(V,\mathbf{F})) & \overset{H^s(P,\delta)}{\rightarrow} & H^2(P, H^{t+1}(V,\mathbf{F})) \\ \downarrow d_2^{s,t} & & \downarrow d_2^{s,t+1} \\ H^{s+2}(P, H^{t-1}(V,\mathbf{F})) & \overset{H^{s+2}(P,\delta)}{\rightarrow} & H^{s+2}(P, H^t(V,\mathbf{F})) \end{array}$$

(*By abuse of notation $d_2\delta + \delta d_2 = 0$.*)

PROOF. We prove a somewhat more general result. Let A be a $\mathbf{Z}$-free double complex and put $\bar{A} = A/pA$. (In particular, consider $A = \operatorname{Hom}_P(Y, \operatorname{Hom}_V(X,\mathbf{Z}))$.) Let $\alpha \in H_{\mathrm{II}}^s(H_{\mathrm{I}}^*(\bar{A}))$ be represented by $\bar{a} + \bar{b}$ where $\bar{a} \in \bar{A}^{s,t}$, $\bar{b} \in \bar{A}^{s+1,t-1}$, $d_{\mathrm{I}}\bar{a} = 0$, $d_{\mathrm{II}}\bar{a} + d_{\mathrm{I}}\bar{b} = 0$. Then $d_2\alpha$ is represented by$d(\bar{a}+\bar{b}) = d_{\mathrm{II}}\bar{b}$. Let $a \in A^{s,t}$, $b \in A^{s+1,t-1}$ represent $\bar{a}$ and $\bar{b}$ respectively. Then $d_{\mathrm{I}}a = pg$ for $g \in A^{s,t+1}$, and $d_{\mathrm{II}}a + d_{\mathrm{I}}b - ph = 0$ for $h \in A^{s+1,t}$. Hence, $d_{\mathrm{II}}d_{\mathrm{I}}a = pd_{\mathrm{II}}g$, and $d_{\mathrm{I}}d_{\mathrm{II}}a - pd_{\mathrm{I}}h = 0$; so $-d_{\mathrm{II}}d_{\mathrm{I}}a - pd_{\mathrm{I}}h = 0$, and $p(d_{\mathrm{II}}g + d_{\mathrm{I}}h) = 0$. Since A is $\mathbf{Z}$-free $d_{\mathrm{II}}g + d_{\mathrm{I}}h = 0$. Clearly, $\bar{g} + \bar{h}$ represents $\delta\alpha \in H_{\mathrm{II}}^s(H_{\mathrm{I}}^{t+1}(\bar{A}))$, and $d_{\mathrm{II}}\bar{h}$ represents $d_2\delta\alpha$. However, from $d_{\mathrm{II}}a + d_{\mathrm{I}}b - ph = 0$, we also obtain $d_{\mathrm{II}}d_{\mathrm{I}}b = pd_{\mathrm{II}}h$ or $-d_{\mathrm{I}}d_{\mathrm{II}}b = pd_{\mathrm{II}}h$. Since $d_{\mathrm{II}}b \in A^{s+2,t-1}$ represents $d_{\mathrm{II}}\bar{b} \in \bar{A}^{s+2,t-1}$ (representing $d_2\alpha$), it follows that $-d_{\mathrm{II}}\bar{h}$ represents $\delta d_2\alpha$ as required. □

PROPOSITION 11.6. *For $p > 3$,*

$$d_2^{0,3}(\alpha_1\delta\alpha_1) = 0, \qquad d_2^{0,3}(\alpha_1\delta\alpha_2 - \alpha_2\delta\alpha_1) = 0.$$

PROOF. The first result follows since E_2 is a differential *ring* and $d_2\alpha_1 = d_2\delta\alpha_1 = 0$. The second follows by Lemma 11.5 since $\delta(\alpha_1\alpha_2) = (\delta\alpha_1)\alpha_2 - \alpha_1(\delta\alpha_2)$ and $d_2(\alpha_1\alpha_2) = 0$. □

PROPOSITION 11.7. *For* $p > 3$,

$$d_2^{0,3}(\alpha_1\alpha_2\alpha_3) = 0.$$

PROOF.

$$\begin{aligned}(y-1)(a_1a_2a_3) &= a_1(a_1+a_2)(a_2+a_3) - a_1a_2a_3\\ &= a_1^2a_2 + a_1^2a_3 + a_1a_2^2.\end{aligned}$$

However, $a_k^2 = d_V u_k$ where $u_k(v) = -\frac{1}{2}i_k^2$ for $v = i_1x_1 + i_2x_2 + i_3x_3$. Hence,

$$\begin{aligned}(y-1)(a_1a_2a_3) &= (d_Vu_1)a_2 + (d_Vu_1)a_3 + a_1d_Vu_2\\ &= d_V(u_1a_2 + u_1a_3 - a_1u_2).\end{aligned}$$

Call the expression in parentheses c. Since $yu_1 = u_1$, $ya_1 = a_1$, we have $Tc = u_1Ta_2 + u_1Ta_3 - a_1Tu_2$. Because $T\hat{V} = 0$, $Ta_2 = Ta_3 = 0$, and $Tc = -a_1Tu_2$. From (9.2), we see that $(yu_2)(v) = -\frac{1}{2}(i_1+i_2)^2$. Hence,

$$\begin{aligned}(Tu_2)(v) &= -\frac{1}{2}\sum_{j=0}^{p-1}(ji_1+i_2)^2\\ &= -\frac{1}{2}\left[\left(\sum_{j=0}^{p-1} j^2\right)i_1 + 2\left(\sum_{j=0}^{p-1} j\right)i_1i_2 + pi_2^2\right] = 0 \quad \text{for } p > 3.\end{aligned}$$

Thus $Tc = 0$ as required. □

PROPOSITION 11.8. *For* $p > 3$,

$$d_2^{0,3}(\alpha_1\delta\alpha_3 + \alpha_3\delta\alpha_1 - \alpha_2\delta\alpha_2 + \alpha_2\delta\alpha_1) = 2\alpha_1\alpha_2.$$

PROOF. Recall that $\delta\alpha_k$ is represented by δa_k defined by

$$\delta a_k(v,w) = \begin{cases} 0, & 0 \leqslant i_k + j_k < p,\\ 1, & p \leqslant i_k + j_k < 2p,\end{cases}$$

where $v = i_1x_1 + i_2x_2 + i_3x_3$, $w = j_1x_1 + j_2x_2 + j_3x_3$, $0 \leqslant i_k, j_k < p$.

We introduce some notation from computer programming. If $k = pq + r$, $0 \leqslant r < p$, write $k/p = q$ and $\mathrm{mod}(k) = r$. Then, $\delta a_k(v,w) = (i_k + j_k)/p$. ($k/p$ and $\mathrm{mod}(k)$ are integers, but we shall abuse notation by viewing them in $\mathbf{Z}/p\mathbf{Z} = \mathbf{F}$. The context will make clear which is meant.)

LEMMA 11.9. *For* $k = 2, 3$,

$$y\delta a_k = \delta a_{k-1} + \delta a_k + d_Vf_k$$

where $f_k(v) = -(i_{k-1} + i_k)/p$. *Also*, $y\delta a_1 = \delta a_1$.

PROOF. We do the case $k = 2$. From (9.2), $y^{-1}(v) = i_1x_1 + (i_1 + i_2)x_2 + (i_2 + i_3)x_3$. Hence, $(y\delta a_2)(v, w) = \delta a_2(v', w')$ where

$$v' = i_1x_1 + \operatorname{mod}(i_1 + i_2)x_2 + \operatorname{mod}(i_2 + i_3)x_3,$$
$$w' = j_1x_1 + \operatorname{mod}(j_1 + j_2)x_2 + \operatorname{mod}(j_2 + j_3)x_3.$$

Hence, $(y\delta a_2)(v, w) = (\operatorname{mod}(i_1 + i_2) + \operatorname{mod}(j_1 + j_2))/p$. Hence,

$$(y\delta a_2 - \delta a_1 - \delta a_2)(v, w) = (*)$$

where

$$(*) = \big(\operatorname{mod}(i_1 + i_2) + \operatorname{mod}(j_1 + j_2)\big)/p - (i_1 + j_1)/p - (i_2 + j_2)/p.$$

On the other hand, $d_V f_2(v, w) = (**)$ where

$$(**) = -(j_1 + j_2)/p + \big(\operatorname{mod}(i_1 + j_1) + \operatorname{mod}(i_2 + j_2)\big)/p - (i_1 + i_2)/p.$$

However,

$$i_1 + j_1 = p\big((i_1 + j_1)/p\big) + \operatorname{mod}(i_1 + j_1),$$
$$i_2 + j_2 = p\big((i_2 + j_2)/p\big) + \operatorname{mod}(i_2 + j_2),$$

so,

$$i_1 + j_1 + i_2 + j_2 = p\big[(i_1 + j_1)/p + (i_2 + j_2)/p\big] + \operatorname{mod}(i_1 + j_1) + \operatorname{mod}(i_2 + j_2),$$

and

$$(i_1 + j_1)/p + (i_2 + j_2)/p = (i_1 + j_1 + i_2 + j_2)/p - \big[\operatorname{mod}(i_1 + j_1) + \operatorname{mod}(i_2 + j_2)\big]/p.$$

Hence,

$$(*) = \big[\operatorname{mod}(i_1 + i_2) + \operatorname{mod}(j_1 + j_2)\big]/p + \big[\operatorname{mod}(i_1 + j_1) + \operatorname{mod}(i_2 + j_2)\big]/p - (i_1 + j_1 + i_2 + j_2)/p.$$

Similarly,

$$(i_1 + i_2)/p + (j_1 + j_2)/p = (i_1 + i_2 + j_1 + j_2)/p - \big(\operatorname{mod}(i_1 + i_2) + \operatorname{mod}(j_1 + j_2)\big)/p,$$

so

$$(**) = \big(\operatorname{mod}(i_1 + j_1) + \operatorname{mod}(i_2 + j_2)\big)/p + \big(\operatorname{mod}(i_1 + i_2) + \operatorname{mod}(j_1 + j_2)\big)/p - (i_1 + i_2 + j_1 + j_2)/p.$$

Hence, $(**) = (*)$ as required. □

To continue the proof of the proposition, calculate as follows using (11.3).

$$\begin{aligned}
&(y - 1)(a_1\delta a_3 + a_3\delta a_1 - a_2\delta a_2 + a_2\delta a_1)\\
&\quad = a_1(\delta a_2 + \delta a_3 + d_V f_3) + (a_2 + a_3)\delta a_1 - (a_1 + a_2)(\delta a_1 + \delta a_2 + d_V f_2)\\
&\qquad + (a_1 + a_2)\delta a_1 - a_1\delta a_3 - a_3\delta a_1 + a_2\delta a_2 - a_2\delta a_1\\
&\quad = a_1 d_V f_3 - (a_1 + a_2)d_V f_2\\
&\quad = d_V(-a_1 f_3 + (a_1 + a_2)f_2).
\end{aligned}$$

We need to calculate Tc where $c = (a_1 + a_2)f_2 - a_1 f_3$. We have

$$\begin{aligned} Tc &= T(a_1 f_2) + T(a_2 f_2) - T(a_1 f_3) \\ &= a_1 Tf_2 + T(a_2 f_2) - a_1 Tf_3. \end{aligned}$$

However,

$$\begin{aligned} T(a_2 f_2) &= \sum_{k=0}^{p-1} (y^k a_2)(y^k f_2) = \sum_{k=0}^{p-1} (ka_1 + a_2)(y^k f_2) \\ &= a_1 \left(\sum_{k=0}^{p-1} k y^k f_2 \right) + a_2(Tf_2). \end{aligned}$$

Thus, $Tc = (a_1 + a_2)Tf_2 + a_1(T'f_2 - Tf_3)$ where $T'f_2 = \Sigma_{k=0}^{p-1} k y^k f_2$. We shall show that $Tf_2 = -a_1$ and $T'f_2 - Tf_3 = \frac{1}{2} a_1 + a_2$. From this it follows that

$$\begin{aligned} Tc &= -(a_1 + a_2)a_1 + a_1(\tfrac{1}{2}a_1 + a_2) \\ &= a_1 a_2 - a_2 a_1 - \tfrac{1}{2}a_1^2 = a_1 a_2 - a_2 a_1 - \tfrac{1}{2} d_V u_1. \end{aligned}$$

Thus, Tc represents $\alpha_1\alpha_2 - \alpha_2\alpha_1 = 2\alpha_1\alpha_2$ as required.

Thus, we need only establish

LEMMA 11.10. $Tf_2 = -a_1$, $T'f_2 - Tf_3 = \frac{1}{2}a_1 + a_2$.

PROOF. Define $i_2^{(k)}$, $k = 0, 1, \ldots, p$, inductively by $i_2^{(0)} = i_2$, $i_1 + i_2^{(k)} = pq_k + i_2^{(k+1)}$, $0 \leq i_2^{(k+1)} < p$. ($i_2^{(k)}$ is the coefficient of x_2 in $y^{-k}v$ for $v = i_1x_1 + i_2x_2 + i_3x_3$.)

Then adding yields

$$(\dagger) \qquad p i_1 + \sum_{k=0}^{p-1} i_{2(k)} = p \sum_{k=0}^{p-1} q_k + \sum_{k=1}^{p} i_{2(k)}.$$

From this we conclude $i_2^{(p)} \equiv i_2^{(0)} \bmod p$ so $i_2^{(p)} = i_2$. Again, from (†) we get $\Sigma_{k=0}^{p-1} q_k = i_1$. However, $(y^k f_2)(v) = f_2(y^{-k}v) = -(i_1 + i_2^{(k)})/p = -q_k$ so $(Tf_2)(v) = -\Sigma_{k=0}^{p-1} q_k = -i_1 = -a_1(v)$, and $Tf_2 = -a_1$.

Furthermore, multiplying $i_1 + i_2^{(k)} = pq_k + i_2^{(k+1)}$ by k and adding yields

$$\left(\sum_{k=0}^{p-1} k \right) i_1 + \sum_{k=0}^{p-1} k i_2^{(k)} = p \sum_{k=0}^{p-1} k q_k + \sum_{k=0}^{p-1} k i_2^{(k+1)}.$$

But,

$$\begin{aligned} \sum_{k=0}^{p-1} k i_2^{(k+1)} &= \sum_{k=1}^{p} (k-1) i_2^{(k)} = \sum_{k=1}^{p} k i_2^{(k)} - \sum_{k=1}^{p} i_2^{(k)} \\ &= \sum_{k=0}^{p-1} k i_2^{(k)} + p i_2 - \sum_{k=0}^{p-1} i_2^{(k)} \end{aligned}$$

(use $i_2^{(p)} = i_2^{(0)} = i_2$). Putting this in the above equation yields

$$\frac{p(p-1)}{2} i_1 = p \sum_{k=0}^{p-1} k q_k + p i_2 - \sum_{k=0}^{p-1} i_2^{(k)}.$$

Hence,

$$(T'f_2)(v) = \sum_{k=0}^{p-1} kf_2(y^{-k}v) = -\sum_{k=0}^{p-1} kq_k$$
$$= -\frac{p-1}{2}i_1 + i_2 - \frac{1}{p}\left(\sum_{k=0}^{p-1} i_2^{(k)}\right).$$

To calculate Tf_3, define $i_3^{(k)}$ inductively by $i_3^{(0)} = i_3$, $i_2^{(k)} + i_3^{(k)} = pr_k + i_3^{(k+1)}$, $0 \leqslant i_3^{(k+1)} < p$. ($i_3^{(k)}$ is the coefficient of x_3 in $y^{-k}v$ for $v = i_1x_1 + i_2x_2 + i_3x_3$.) Adding yields

$$\sum_{k=0}^{p-1} i_2^{(k)} + \sum_{k=0}^{p-1} i_3^{(k)} = p\sum_{k=0}^{p-1} r_k + \sum_{k=1}^{p} i_3^{(k)}.$$

If $i_1 = 0$, $\sum_{k=0}^{p-1} i_2^{(k)} = pi_2$; if $i_1 \neq 0$, $i_2^{(0)}, i_2^{(1)}, \ldots, i_2^{(p-1)}$ is a permutation of $0, 1, 2, \ldots, p-1$, and $\sum_{k=0}^{p-1} i_2^{(k)} = p(p-1)/2$. In either case, the sum is $\equiv 0 \bmod p$. As above, it follows that $i_3^{(p)} = i_3^{(0)} = i_3$, and

$$(Tf_3)(v) = -\sum_{k=0}^{p-1} r_k = -\frac{1}{p}\sum_{k=0}^{p-1} i_2^{(k)}.$$

Hence,

$$(T'f_2)(v) - (Tf_3)(v) = -\frac{(p-1)}{2}i_1 + i_2 = \frac{1}{2}i_1 + i_2.$$

Thus $T'f_2 - Tf_3 = \frac{1}{2}a_1 + a_2$ as claimed. □

12. d^2 for GL(2, p). We may summarize the results of §§10 and 11 in a table.

PROPOSITION 12.1. *For $p > 3$, $d_2^{0,t}\xi$ is given in Table 2 for $t = 1, 2, 3$.*

TABLE 2

ξ	*Split case*	*Nonsplit case* $\overline{U}$
α_1	0	0
$\alpha_1\alpha_2$	0	0
$\delta\alpha_1$	0	0
$\alpha_1\alpha_2\alpha_3$	0	$-\frac{1}{2}\alpha_1\alpha_2$
$\alpha_1\delta\alpha_1$	0	0
$\alpha_1\delta\alpha_2 - \alpha_2\delta\alpha_1$	0	0
$\alpha_1\delta\alpha_3 + \alpha_3\delta\alpha_1 - \alpha_2\delta\alpha_2 + \alpha_2\delta\alpha_1$	$2\alpha_1\alpha_2$	$2\alpha_1\alpha_2 - \frac{1}{2}\delta\alpha_1$

We now make the transition to GL(2, p) and then to homology. We need only concern ourselves with $t = 3$. Using additive notation, we have $z_1 = e_{21}$, $z_2 = h_1$, and $z_3 = e_{12}$ and we denote the dual basis by e^{21}, h^1, and e^{12}. Referring to §9, we see easily that $e^{21} = \alpha_1$, $h^1 = \alpha_2$, $e^{12} = -\alpha_2 - 2\alpha_3$. We know (by Proposition 3.2) that $H^0(\mathrm{GL}(2, p), H^3(V, \mathbf{F}))$ (dual to $H_0(\mathrm{GL}(2, p), H_3(V, \mathbf{F}))$) is 2 dimensional, and it is easy to see that it is the subspace of $H^3(V, \mathbf{F})^P$ spanned by $\chi_1 = e^{21}h^1e^{12} = -2\alpha_1\alpha_2\alpha_3$ and $\chi_2 = e^{21}\delta e^{12} + e^{12}\delta e^{21} + 2h^1\delta h^1 = -\alpha_1\delta\alpha_2 - 2\alpha_1\delta\alpha_3 - \alpha_2\delta\alpha_1 - 2\alpha_3\delta\alpha_1 + 2\alpha_2\delta\alpha_2$ or

$$\chi_2 = -2(\alpha_1\delta\alpha_3 + \alpha_3\delta\alpha_1 - \alpha_2\delta\alpha_2 + \alpha_2\delta\alpha_1) - (\alpha_1\delta\alpha_2 - \alpha_2\delta\alpha_1).$$

From Table 2, we have

PROPOSITION 12.2. *For* $p > 3$, $d_2^{0,3}(\xi)$ *is given by*

ξ	*Split case*	*Nonsplit case*
χ_1	0	$\alpha_1\alpha_2$
χ_2	$-4\alpha_1\alpha_2$	$-4\alpha_1\alpha_2 + \delta\alpha_1$

Note. $d_2(\xi) \in H^2(\mathrm{GL}(2, p), H^2(V, \mathbf{F}))$ but we may identify this group with $H^2(P, H^2(V, \mathbf{F})) = (\wedge^2 \hat{V})^P \oplus \hat{V}^P$ through restriction. For, by Proposition 5.2,

$$H^2(\mathrm{GL}(2, p), \hat{V}_2) = \widehat{H_2(\mathrm{GL}(2, p), V)} = \mathbf{Z}/p\mathbf{Z},$$

so res: $H^2(\mathrm{GL}(2, p), \hat{V}) \to H^2(P, \hat{V}) = \mathbf{Z}/p\mathbf{Z}$ is an isomorphism. Since $V \wedge V \cong V$ also, the above contention follows.

It now follows that $d_2^{0,3}$: $H^0(\mathrm{GL}(2, p), H^3(V_2, \mathbf{F})) \to H^2(\mathrm{GL}(2, p), H^2(V_2, \mathbf{F}))$ has rank one in the split case and is an isomorphism in the nonsplit case. By duality, we have

PROPOSITION 12.3. *For* $p > 3$, *we have that* (a) $d_{2,2}^2$: $H_2(\mathrm{GL}(2, p), H_2(V_2, \mathbf{Z}/p\mathbf{Z})) \to H_0(\mathrm{GL}(2, p), H_3(V_2, \mathbf{Z}/p\mathbf{Z}))$ *is an isomorphism for the nonsplit extension* $1 \to V_2 \to \overline{\mathrm{SL}}(2, p^2) \to \mathrm{GL}(2, p) \to 1$ *and*

(b) $d_{2,2}^2$ *has rank* 1 *for the corresponding split extension.*

To deal with integral homology, we consider dually the coefficient group $\mathbf{Q}/\mathbf{Z}$. The exact sequence $0 \to \mathbf{Z}/p\mathbf{Z} \to \mathbf{Q}/\mathbf{Z} \xrightarrow{p} \mathbf{Q}/\mathbf{Z} \to 0$ yields

$$0 \to H^2(V_2, \mathbf{Q}/\mathbf{Z}) \to H^3(V_2, \mathbf{Z}/p\mathbf{Z}) \to H^3(V_2, \mathbf{Q}/\mathbf{Z}) \to 0.$$

Since

$$H^r(\mathrm{GL}(2, p), H^2(V_2, \mathbf{Q}/\mathbf{Z})) \cong \widehat{H_r(\mathrm{GL}(2, p), H_2(V_2, \mathbf{Z}))} = 0$$

for $r = 0, 1$ (Proposition 5.2 and $V_2 \wedge V_2 \cong V_2$), we have

$$H^0(\mathrm{GL}(2, p), H^3(V_2, \mathbf{Z}/p\mathbf{Z})) \underset{\to}{\cong} H^0(\mathrm{GL}(2, p), H^3(V_2, \mathbf{Q}/\mathbf{Z})).$$

Similarly, we have $0 \to H^1(V_2, \mathbf{Q}/\mathbf{Z}) \to H^2(V_2, \mathbf{Z}/p\mathbf{Z}) \to H^2(V_2, \mathbf{Q}/\mathbf{Z}) \to 0$, and

$$H^3(\mathrm{GL}(2, p), H^1(V_2, \mathbf{Q}/\mathbf{Z})) \cong \widehat{H_3(\mathrm{GL}(2, p), V_2)} = 0$$

(Proposition 5.2). Hence,

$$H^2(\mathrm{GL}(2, p), H^2(V_2, \mathbf{Z}/p\mathbf{Z})) \to H^2(\mathrm{GL}(2, p), H^2(V_2, \mathbf{Q}/\mathbf{Z}))$$

is an epimorphism. It now follows easily that for the nonsplit extension $d_2^{0,3}$: $H^0(\mathrm{GL}(2, p), H^3(V_2, \mathbf{Q}/\mathbf{Z})) \to H^2(\mathrm{GL}(2, p), H^2(V_2, \mathbf{Q}/\mathbf{Z}))$ is an epimorphism. To deal with the split extension, consider

$$\begin{array}{ccccccccc} 0 & \to & \mathbf{Z}/p\mathbf{Z} & \to & \mathbf{Z}/p^2\mathbf{Z} & \to & \mathbf{Z}/p\mathbf{Z} & \to & 0 \\ & & \| & & \| & & \| & & \\ 0 & \to & \mathbf{Z}/p\mathbf{Z} & \to & \mathbf{Q}/\mathbf{Z} & \xrightarrow{p} & \mathbf{Q}/\mathbf{Z} & \to & 0. \end{array}$$

It follows that $\delta\alpha_1 \in H^2(V_2, \mathbf{Z}/p\mathbf{Z})$ is in the image of $H^1(V_2, \mathbf{Q}/\mathbf{Z})$; hence, $\delta\alpha_1 \mapsto 0$ in $H^2(V_2, \mathbf{Q}/\mathbf{Z})$. Thus, under the epimorphism

$$H^2\big(\mathrm{GL}(2, p), H^2(V_2, \mathbf{Z}/p\mathbf{Z})\big) \to H^2\big(\mathrm{GL}(2, p), H^2(V_2, \mathbf{Q}/\mathbf{Z})\big),$$

the image of $\alpha_1\alpha_2$ (viewed as an element on the left through res^{-1}) spans $H^2(\mathrm{GL}(2, p), H^2(V_2, \mathbf{Q}/\mathbf{Z}))$. Since $d_2^{0,3}\chi_2 = -4\alpha_1\alpha_2$, it follows that $d_2^{0,3}$ is an epimorphism for $\mathbf{Q}/\mathbf{Z}$. By duality, we have

PROPOSITION 12.4. *For $p > 3$, we have for both* (a) *split and* (b) *nonsplit extensions,*

$$d_{2,2}^2 \colon H_2\big(\mathrm{GL}(2, p), H_2(V_2, \mathbf{Z})\big) \to H_0\big(\mathrm{GL}(2, p), H_3(V_2, \mathbf{Z})\big)$$

is a monomorphism (*so its image has dimension* 1).

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